The hybrid mean value of the Smarandache function and the Mangoldt function

Baohuai Shi

Department of Mathematics and Physics, Shaanxi Institute of Education
Xi’an, Shaanxi, P.R.China

Abstract For any positive integer \( n \), the famous F.Smarandache function \( S(n) \) is defined as the smallest positive integer \( m \) such that \( n| m! \). The main purpose of this paper is using the elementary methods to study the hybrid mean value of the Smarandache function \( S(n) \) and the Mangoldt function \( \Lambda(n) \), and prove an interesting hybrid mean value formula for \( S(n) \Lambda(n) \).

Keywords F. Smarandache function, Mangoldt function, hybrid mean value, asymptotic formula

§1. Introduction

For any positive integer \( n \), the famous F.Smarandache function \( S(n) \) is defined as the smallest positive integer \( m \) such that \( n| m! \). That is, \( S(n) = \min \{ m : n|m!, m \in \mathbb{N} \} \). From the definition of \( S(n) \) one can easily deduce that if \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) is the factorization of \( n \) into prime powers, then \( S(n) = \max_{1 \leq i \leq k} \{ S(p_i^{a_i}) \} \). From this formula we can easily get: \( S(1) = 1, S(2) = 2, S(3) = 3, S(4) = 4, S(5) = 5, S(6) = 3, S(7) = 7, S(8) = 4, S(9) = 6, S(10) = 5, S(11) = 11, S(12) = 4, S(13) = 13, S(14) = 7, S(15) = 5, S(16) = 6, \cdots \). About the elementary properties of \( S(n) \), many people had studied it, and obtained some important results. For example, Wang Yongxing [2] studied the mean value properties of \( S(n) \), and obtained that:

\[
\sum_{n \leq x} S(n) = \frac{\pi^2}{12} x^2 \ln x + O \left( \frac{x^2}{\ln^2 x} \right).
\]

Lu Yaming [3] studied the positive integer solutions of an equation involving the function \( S(n) \), and proved that for any positive integer \( k \geq 2 \), the equation

\[
S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)
\]

has infinity positive integer solutions \((m_1, m_2, \cdots, m_k)\).

Jozsef Sandor [4] obtained some inequalities involving the F.Smarandache function. That is, he proved that for any positive integer \( k \geq 2 \), there exists infinite positive integer \((m_1, m_2, \cdots, m_k)\) such that the inequalities

\[
S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).
\]
Vol. 2 The hybrid mean value of the Smarandache function and the Mangoldt function

\[(m_1, m_2, \cdots, m_k) \text{ such that } S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).\]

On the other hand, Dr. Xu Zhefeng [5] proved: Let \(P(n)\) denotes the largest prime divisor of \(n\), then for any real number \(x > 1\), we have the asymptotic formula
\[
\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right)x^\frac{3}{2}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln x}\right),
\]
where \(\zeta(s)\) denotes the Riemann zeta-function.

The main purpose of this paper is using the elementary methods to study the hybrid mean value of the Smarandache function \(S(n)\) and the Mangoldt function \(\Lambda(n)\), which defined as follows:
\[
\Lambda(n) = \begin{cases} 
\ln p, & \text{if } n = p^\alpha, \ p \text{ be a prime, } \alpha \text{ be any positive integer}; \\
0, & \text{otherwise.}
\end{cases}
\]
and prove a sharper mean value formula for \(\Lambda(n) S(n)\). That is, we shall prove the following conclusion:

**Theorem.** Let \(k\) be any fixed positive integer. Then for any real number \(x > 1\), we have
\[
\sum_{n \leq x} \Lambda(n) S(n) = x^2 \cdot \sum_{i=0}^{k} c_i \cdot \frac{x^i}{\ln^i x} + O\left(\frac{x^{2}}{\ln^{k+1} x}\right),
\]
where \(c_i\) \((i = 0, 1, 2, \cdots, k)\) are constants, and \(c_0 = 1\).

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. In fact from the definition of \(\Lambda(n)\) we have
\[
\sum_{n \leq x} \Lambda(n) S(n) = \sum_{\alpha \leq \frac{\ln x}{\ln x}} \sum_{p \leq x^\frac{1}{\alpha}} \Lambda(p^\alpha) S(p^\alpha) = \sum_{\alpha \leq \frac{\ln x}{\ln x}} \sum_{p \leq x^\frac{1}{\alpha}} S(p^\alpha) \ln p
\]
\[
= \sum_{p \leq x} p \cdot \ln p + \sum_{2 \leq \alpha \leq \frac{\ln x}{\ln x}} \sum_{p \leq x^\frac{1}{\alpha}} S(p^\alpha) \ln p. \quad (1)
\]
For any positive integer \(k\), from the prime theorem we know that
\[
\pi(x) = \sum_{p \leq x} 1 = x \cdot \sum_{i=1}^{k} a_i \cdot \frac{x^{i}}{\ln^i x} + O\left(\frac{x^{1}}{\ln^{k+1} x}\right), \quad (2)
\]
where \(a_i\) \((i = 1, 2, \cdots, k)\) are constants, and \(a_1 = 1\).
From the Abel’s identity (see [6] Theorem 4.2) and (2) we have

\[
\sum_{p \leq x} p \cdot \ln p \cdot x \cdot \ln x = \pi(x) \cdot x \cdot \ln x - \int_{2}^{x} \pi(y)(\ln y + 1)dy
\]

\[
= x \ln x \cdot x \cdot \left( \sum_{i=1}^{k} \frac{a_i}{\ln^i x} + O \left( \frac{x}{\ln^{k+1} x} \right) \right) - \int_{2}^{x} \left( \sum_{i=1}^{k} \frac{a_i}{\ln^i y} + O \left( \frac{y}{\ln^{k+1} y} \right) \right) (\ln y + 1)dy
\]

\[
= x^2 \cdot \sum_{i=0}^{k} \frac{c_i}{\ln^i x} + O \left( \frac{x}{\ln^{k+1} x} \right), \tag{3}
\]

where \( c_i \ (i = 0, 1, 2, \cdots, k) \) are constants, and \( c_0 = 1 \).

On the other hand, applying the estimate

\[
S(p^\alpha) \ll \alpha \cdot \ln p,
\]

we have

\[
\sum_{2 \leq \alpha \leq \frac{\ln x}{\ln 2}} \sum_{p \leq x^{1/2}} S(p^\alpha) \ln p \ll \sum_{2 \leq \alpha \leq \frac{\ln x}{\ln 2}} \sum_{p \leq x^{1/2}} \alpha \cdot p \cdot \ln p \ll x \cdot \ln^2 x. \tag{4}
\]

Combining (1)-(4) we have

\[
\sum_{n \leq x} \Lambda(n) S(n) = x^2 \cdot \sum_{i=0}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O \left( \frac{x^2}{\ln^{k+1} x} \right),
\]

where \( c_i \ (i = 0, 1, 2, \cdots, k) \) are constants, and \( c_0 = 1 \).

This completes the proof of the theorem.

References