Some inequalities concerning Smarandache’s function

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The objectives of this article are to study the sum $\sum_{d|n} S(d)$ and to find some upper bounds for Smarandache’s function. This sum is proved to satisfy the inequality

$\sum_{d|n} S(d) \leq n$ at most all the composite numbers. Using this inequality, some new upper bounds for Smarandache’s function are found. These bounds improve the well-known inequality $S(n) \leq n$.

1. Introduction

The object that is researched is Smarandache’s function. This function was introduced by Smarandache [1980] as follows:

$$S: \mathbb{N}^* \rightarrow \mathbb{N} \text{ defined by } S(n) = \min\{k \in \mathbb{N} \mid k! = n\} \quad (\forall n \in \mathbb{N}^*).$$

The following main properties are satisfied by $S$:

$$\left(\forall a, b \in \mathbb{N}^*\right) (a, b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}. \quad (2)$$

$$\left(\forall a \in \mathbb{N}^*\right) S(a) \leq a \text{ and } S(a) = a \text{ if } a \text{ is prim.} \quad (3)$$

$$\left(\forall p \in \mathbb{N}^*, p \text{ prime}\right) \left(\forall k \in \mathbb{N}^*\right) S(p^k) \leq p \cdot k. \quad (4)$$

Smarandache’s function has been researched for more than 20 years, and many properties have been found. Inequalities concerning the function $S$ have a central place and many articles have been published [Smarandache, 1980], [Cojocaru, 1997], [Tabirca, 1997], [Tabirca, 1988]. Two important directions can be identified among these inequalities. First direction and the most important is represented by the inequalities concerning directly the function $S$ such as upper and lower bounds. The second direction is given by the inequalities involving sums or products with the function $S$. 

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2. About the sum \( \sum_{d|n} S(d) \)

The aim of this section is to study the sum \( \sum_{d|n} S(d) \).

Let \( SS(n) = \sum_{d|n} S(d) \) denote the above sum. Obviously, this sum satisfies \( SS(n) = \sum_{1 \leq d \leq n} S(d) \). Table 1 presents the values of \( S(n) \) and \( SS(n) \) for \( n<50 \) [Ibstedt, 1997]. From this table, it can be seen that the inequality \( SS(n) \leq n + 2 \) holds for all \( n=1, 2, \ldots, 50 \) and \( n=12 \). Moreover, if \( n \) is a prime number, then the inequality becomes equality \( SS(n) = n \).

Remarks 1.

a) If \( n \) is a prime number, then \( SS(n) = S(1) + S(n) = n \).

b) If \( n>2 \) is a prim number, then
\[
SS(2 \cdot n) = S(1) + S(2) + S(n) + S(2 \cdot n) = 2 + n + n = 2 \cdot n + 2 ,
\]
c) \( SS(n^2) = S(1) + S(n) + S(n^2) = n + 2 \cdot n = 3 \cdot n \leq n^2 \).

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Table 1. The values of \( n, S, SS \).
The inequality $SS(n) \leq n$ is proved to be true for the following particular values $n = p^k, 2 \cdot p^k, 3 \cdot p^k$ and $6 \cdot p^k$.

**Lemma 1.** If $p > 2$ is a prime number and $k > 1$, then the inequality $SS(p^k) \leq p^k$ holds.

**Proof**

The following inequality holds according to inequality (4) and the definition of $SS$.

$$SS(p^k) = \sum_{i=1}^{k} S(p^i) \leq \sum_{i=1}^{k} p \cdot i = p \cdot \frac{k \cdot (k + 1)}{2}.$$

The inequality

$$\sum_{i=1}^{k} p \cdot i = p \cdot \frac{k \cdot (k + 1)}{2} \leq p^k$$

is proved to be true by analysing the following cases.

- $k = 2 \Rightarrow 3 \cdot p \leq p^2$. (6)
- $k = 3 \Rightarrow 6 \cdot p \leq p^3$. (7)
- $k = 4 \Rightarrow 10 \cdot p \leq p^4$. (8)

Inequalities (6-8) are true because $p > 2$.

- $k = 4 \Rightarrow p^k \geq p \cdot p^{k-1} \geq p \cdot 2^{k-1} = p \cdot \sum_{i=0}^{k-1} (k - 1)$. The first and the last three terms of this sum are kept and it is found

$$p^k \geq p \cdot \left(2 \cdot \binom{k-1}{0} + 2 \cdot \binom{k-1}{1} + 2 \cdot \binom{k-1}{2}\right) = p \cdot (k^2 - k + 2) .$$

The inequality

$$k^2 - k + 2 \geq \frac{k \cdot (k + 1)}{2}$$

holds because $k > 4$, therefore $p^k \geq p \cdot \frac{k \cdot (k + 1)}{2}$ is true.

Therefore, the inequality $S(p^k) \leq p^k$ holds.

**Remark 2.** The inequality $S(p^k) \leq p^k$ is still true for $p = 2$ and $k > 3$ because (8) holds for these values. Table 1 shows that the inequality is not true for $p = 2$ and $k = 2, 3$.

**Lemma 2.** If $p > 2$ is a prime number and $k > 1$, then the inequality $SS(2 \cdot p^k) \leq 2 \cdot p^k$ holds.
Proof

The definition of $SS$ gives the following equation

$$SS(p^k) = S(2) + \sum_{i=1}^{k} S(p^i) + \sum_{i=1}^{k} S(2 \cdot p^i).$$

Applying the inequality $S(2 \cdot p^i) \leq p \cdot i$ and (4), we have

$$SS(2 \cdot p^k) \leq 2 + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i = 2 + p \cdot k \cdot (k + 1).$$

(9)

The inequality

$$2 + p \cdot k \cdot (k + 1) \leq 2 \cdot p^k$$

is proved to be true as before.

- $k=2 \Rightarrow 2 + 6 \cdot p \leq 2 \cdot p^2$. (11)
- $k=3 \Rightarrow 2 + 12 \cdot p \leq 2 \cdot p^3$. (12)
- $k=4 \Rightarrow 2 + 20 \cdot p \leq 2 \cdot p^4$. (13)
- $k=5 \Rightarrow 2 + 30 \cdot p \leq 2 \cdot p^5$. (14)
- $k=6 \Rightarrow 2 + 42 \cdot p \leq 2 \cdot p^6$. (15)

These above inequalities (11-15) are true because $p > 2$.

- $k \geq 6 \Rightarrow p^k \geq p \cdot p^{k-1} \geq p \cdot 2^{k-1} = p \cdot \sum_{i=0}^{k-1} \binom{k-1}{i}$. The first and the last fourth terms of this sum are kept finding

$$p^k \geq p \cdot \left(2 \cdot \binom{k-1}{0} + 2 \cdot \binom{k-1}{1} + 2 \cdot \binom{k-1}{2} + 2 \cdot \binom{k-1}{3}\right) \geq$$

$$\geq p \cdot \left(2 \cdot 1 + 2 \cdot \binom{k-1}{1} + 2 \cdot \binom{k-1}{2} + 2 \cdot \binom{k-1}{3}\right) =$$

$$= p \cdot (2 \cdot k^2 - 4 \cdot k + 4) \geq 2 + p \cdot (k^2 + k)$$

The last inequality holds because $k \geq 6$, therefore $2 \cdot p^k \geq 2 + p \cdot k \cdot (k + 1)$ is true.

The inequality $SS(2 \cdot p^k) \leq 2 \cdot p^k$ holds because (10) has been found to be true.

Remark 3. Similarly, the inequality $SS(3 \cdot p^k) \leq 3 \cdot p^k$ can be proved for all ($p > 3$ and $k \geq 1$) or ($p = 2$ and $k \geq 3$).
**Lemma 3.** If \( p > 3 \) is a prime number and \( k \geq 1 \), then the inequality \( SS(6 \cdot p^k) \leq 6 \cdot p^k \) holds.

**Proof**

The starting point is given by the following equation (16)

\[
SS(6 \cdot p^k) = S(2) + S(3) + S(6) + \sum_{i=1}^{k} S(p^i) + \sum_{i=1}^{k} S(2 \cdot p^i) + \sum_{i=1}^{k} S(3 \cdot p^i) + \sum_{i=1}^{k} S(6 \cdot p^i).
\]

The inequalities \( S(p^i), S(2 \cdot p^i), S(3 \cdot p^i), S(6 \cdot p^i) \leq p \cdot i \) hold for all \( i > 1 \) because \( p > 5 \). Therefore, the inequality

\[
SS(6 \cdot p^k) \leq 8 + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i = 8 + 4 \cdot \sum_{i=1}^{k} p \cdot i
\]

holds. The inequality \( SS(6 \cdot p^k) \leq 8 + 4 \cdot p^k \leq 6 \cdot p^k \) is found to be true by applying (5) in (17).

\[\Box\]

The following propositions give the main properties of the function \( SS \). Let \( d(n) \) denote the number of divisors of \( n \).

**Proposition 1.** If \( a \) is a natural number such that \( S(a) \geq 4 \), then the inequality \( S(a) \geq 2 \cdot d(a) \) holds.

**Proof**

The proof is made directly as follows:

\[
S(a) = \sum_{d \mid a} S(d) = \sum_{d \mid a} S(d) + S(a) \geq \sum_{d \mid a} 2 + S(a) = 2 \cdot (d(a) - 2) + S(a) = 2 \cdot d(a) + S(a) - 4 \geq 2 \cdot d(a).
\]

\[\Box\]

**Remark 4.** The inequality \( S(a) \geq 4 \) is verified for all the numbers \( a \geq 4 \) and \( a \neq 6 \).

**Proposition 2.** If \( a, b \) are two natural numbers such that \( (a, b) = 1 \), then the inequality \( SS(a \cdot b) \leq d(a) \cdot SS(b) + d(b) \cdot SS(a) \) holds.

**Proof**

This proof is made by using (2) and the simple remark that \( a, b \geq 0 \Rightarrow \max\{a, b\} \leq a + b \).
The set of the divisors of $ab$ is split into three sets as follows:

$$\{l \neq d \mid a \cdot b = Md\} = \{l \neq d \mid a = Md\} \cup \{l \neq d \mid b = Md\} \cup \{d, d_2 \mid a = Md, 1 \neq b = Md, 1 \land (d, d_2) = 1\}. \quad (18)$$

The following transformations hold according to (18).

$$SS(a \cdot b) = \sum_{(1 \neq d \mid a \cdot b \neq Md)} S(d) = \sum_{(1 \neq d \mid a \cdot b \neq Md)} S(d) + \sum_{(1 \neq d_1 \mid a \cdot b \neq Md)} S(d_1) + \sum_{(1 \neq d_2 \mid b \neq Md)} S(d_2) =$$

$$= SS(a) + SS(b) + \sum_{(1 \neq d_1 \mid a \cdot b \neq Md)} \max\{S(d_1), S(d_2)\} \leq$$

$$\leq SS(a) + SS(b) + \sum_{(1 \neq d_1 \mid a \cdot b \neq Md)} [S(d_1) + S(d_2)] =$$

$$= SS(a) + SS(b) + \sum_{(1 \neq d_1 \mid a \cdot b \neq Md)} S(d_1) + \sum_{(1 \neq d_2 \mid b \neq Md)} S(d_2) =$$

$$= SS(a) + SS(b) + SS(a) \cdot [d(b) - 1] + SS(b) \cdot [d(a) - 1]$$

Therefore, the inequality $SS(a \cdot b) \leq d(a) \cdot SS(b) + d(b) \cdot SS(a)$ holds.

**Proposition 3.** If $a, b$ are two natural numbers such that $S(a), S(b) \geq 4$ and $(a, b) = 1$, then the inequality $SS(a \cdot b) \leq SS(a) \cdot SS(b)$ holds.

**Proof**

Proposition 1-2 are applied to prove this proposition as follows:

$$S(a), S(b) \geq 4 \Rightarrow S(a) \geq 2 \cdot d(a) \text{ and } S(b) \geq 2 \cdot d(b) \quad (19)$$

$$(a, b) = 1 \Rightarrow SS(a \cdot b) \leq d(a) \cdot SS(b) + d(b) \cdot SS(a). \quad (20)$$

The proof is completed if the inequality $d(a) \cdot SS(b) + d(b) \cdot SS(a) \leq SS(a) \cdot SS(b)$ is found to be true. This is given by the following equivalence

$$d(a) \cdot SS(b) + d(b) \cdot SS(a) \leq SS(a) \cdot SS(b) \Leftrightarrow$$

$$d(a) \cdot d(b) \leq [SS(a) - d(a)] \cdot [SS(b) - d(b)].$$

This last inequality holds according to (19).

Therefore, the inequality $SS(a \cdot b) \leq SS(a) \cdot SS(b)$ is true.

**Theorem 1.** If $n$ is a natural number such that $n \neq 8, 12, 20$ then

a) $SS(n) = n + 2$ if $(\exists p \text{ prime}) n = 2 \cdot p. \quad (21)$

b) $SS(n) \leq n$, otherwise. \quad (22)
Proof

The proof of this theorem is made by using the induction on \( n \).

Equation (21) is true according to Remark 1.a. Table 1 shows that Equation (22) holds for \( n<51 \) and \( n \neq 8, 12, 20 \). Let \( n>51 \) be a natural number. Let us suppose that Equation (9) is true for all the number \( k \) that satisfies \( k<n \) and \( k \) does not have the form \( k=2p, \, p \) prime. The following cases are analysed:

- **\( n \) is prime** \( \Rightarrow SS(n)=n \), therefore Equation (9) holds.
- **\( n=2p, \, p>2 \) prime** \( \Rightarrow SS(n)=n+2 \), therefore Equation (21) holds.
- **\( (n=2^k \text{ and } k>3) \) or \( (n=p^k \text{ and } k>1) \)** \( \Rightarrow SS(n) \leq n \) according to Lemma 1.
- **\( n=2 \cdot p^k, \, p>2 \) prime number and \( k>1 \)** \( \Rightarrow SS(n) \leq n \) according to Lemma 2.
- **\( n=3 \cdot p^k, \, (p>3 \text{ prime number and } k>1) \) or \( (p=2 \text{ and } k>2) \)** \( \Rightarrow SS(n) \leq n \) according to Remark 3.
- **\( n=6 \cdot p^k, \, p>3 \) prime number and \( k\geq1 \)** \( \Rightarrow SS(n) \leq n \) according to Lemma 3.
- **Otherwise** \( \Rightarrow \) Let \( n=p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s} \), be the prime number decomposition of \( n \) with \( p_1 < p_2 < \cdots < p_s \). We prove that there is a decomposition of \( n=ab, \, (a,b)=1 \) such that \( S(a), S(b)\geq4 \). Let us select \( a=p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s} \) and \( b=p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s} \). It is not difficult to see that this decomposition satisfies the above conditions. The induction’s hypotheses is applied for \( a,b<n \) and the inequalities \( SS(a)\leq a \) and \( SS(b)\leq b \) are obtained. Finally, Proposition 3 gives \( SS(n) = SS(a \cdot b) \leq SS(b) \cdot SS(a) \leq b \cdot a = n \).

We can conclude that the inequality \( SS(n) \leq n-2 \) holds for all the natural number \( n\neq12 \).

Remark 5. The above analysis is necessary to be sure that the decomposition of \( n=ab, \, (a,b)=1, \, S(a), S(b)\geq4 \) exists.

Theorem 1 has some interesting consequences that are presented in the following. These establish new upper bounds for Smarandache’s function.

**Consequence 1.** If \( n > 1 \) is a natural number, then the following inequality

\[
S(n) \leq n + 4 - 2 \cdot d(n) \tag{23}
\]

holds.
Proof
The proof of this inequality is made by using Theorem 1.
Obviously, (23) is true for $n=p$ or $n=2p$, $p$ prime number.
Let $n \neq 8, 12, 20$ be a natural number.
We have the following transformations:

\[
SS(n) = \sum_{d \mid n} S(d) = S(n) + \sum_{d \mid n} S(d) \geq \]

\[
S(n) + 2 \cdot d = S(n) + 2 \cdot (d(n) - 2) = S(n) + 2 \cdot d(n) - 4
\]

Inequality (23) is also satisfied for $n=8, 12, 20$.
Therefore, the inequality $S(n) \leq n + 4 - 2 \cdot d(n)$ holds.

Consequence 2. If $n > 1$ is a natural number, then the following inequality holds

\[
S(n) \leq n + 4 - \min \{p \mid p \text{ is prime and } p \mid n\} \cdot d(n)
\]

Proof
This proof is made similarly to the proof of the previous consequence by using the following strong inequality $S(d) \geq \min \{p \mid p \text{ is prime and } p \mid n\}$.

3. Final Remark
Inequalities (23 - 24) give some generalisations of the well-known inequality $S(n) \leq n$. More important is the fact that these inequalities reflect. When $n$ has many divisors, the value of $n + 4 - \min \{p \mid p \text{ is prime and } p \mid n\} \cdot d(n)$ is small, therefore the value of $S(n)$ is small as well according to Inequality (24). In spite of fact that Inequalities (23 - 24) reflect this situation, we could not say that the upper bounds are the lowest possible. Nevertheless, they offer a better upper bound than the inequality $S(n) \leq n$.

References


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