Inequalities for the polygamma functions with application

Chaoping Chen
Department of Applied Mathmatics, Hennan Polytechnic University
Jiaozuo, Hennan, P. R. China

Abstract We present some inequalities for the polygamma functions. As an application, we give the upper and lower bounds for the expression \( \sum_{k=1}^{n} \frac{1}{z} - \ln n - \gamma \), where \( \gamma = 0.57721\cdots \) is the Euler’s constant.

Keywords Inequality; Polygamma function; Harmonic sequence; Euler’s constant.

§1. Inequalities for the Polygamma Function

The gamma function is usually defined for \( Rez > 0 \) by

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt. \]

The psi or digamma function, the logarithmic derivative of the gamma function and the polygamma functions can be expressed as

\[ \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=0}^{\infty} \left( \frac{1}{1+k} - \frac{1}{z+k} \right), \]

\[ \psi^n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \]

for \( Rez > 0 \) and \( n = 1, 2, \cdots \), where \( \gamma = 0.57721\cdots \) is the Euler’s constant.

M. Merkle [2] established the inequality

\[ \frac{1}{x} + \frac{1}{2x^2} + \sum_{k=1}^{2N} \frac{B_{2k}}{x^{2k+1}} < \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} < \frac{1}{x} + \sum_{k=1}^{2N+1} \frac{B_{2k}}{x^{2k+1}} \]

for all real \( x > 0 \) and all integers \( N \geq 1 \), where \( B_k \) denotes Bernoulli numbers, defined by

\[ \frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} t^j. \]

The first five Bernoulli numbers with even indices are

\[ B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}. \]

\footnote{This work is supported in part by SF of Henan Innovation Talents at University of P. R. China}
The following theorem establishes a more general result.

**Theorem 1.** Let \( m \geq 0 \) and \( n \geq 1 \) be integers, then we have for \( x > 0 \),

\[
\ln x - \frac{1}{2x} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x) < \ln x - \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}}
\] (1)

and

\[
\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}} < (-1)^{n+1} \psi^{(n)}(x) < \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}.
\] (2)

**Proof.** From Binet’s formula [6, p. 103]

\[
\ln \Gamma(x) = \left( x - \frac{1}{2} \right) \ln x - x + \ln \sqrt{2\pi} + \int_0^\infty \left( \frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) e^{-xt} \frac{e^{-t}}{t} dt,
\]

we conclude that

\[
\psi(x) = \ln x - \frac{1}{2x} - \int_0^\infty \left( \frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) e^{-xt} \frac{e^{-t}}{t} dt
\] (3)

and therefore

\[
(-1)^{n+1} \psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \int_0^\infty \left( \frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) \frac{t^{n-1}}{e^t} e^{-xt} dt.
\] (4)

It follows from Problem 154 in Part I, Chapter 4, of [3] that

\[
\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} t^{2j} < \frac{t}{e^t - 1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} t^{2j}
\] (5)

for all integers \( m > 0 \). The inequality (5) can be also found in [4].

From (3) and (5) we conclude (1), and we obtain (2) from (4) and (5). This completes the proof of the theorem 1.

Note that \( \psi(x+1) = \psi(x) + \frac{1}{x} \) (see [1, p. 258]), (1) can be written as

\[
\frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x+1) - \ln x < \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}}
\] (6)

and (2) can be written as

\[
\frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}} < (-1)^{n+1} \psi^{(n)}(x) < \frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}.
\] (7)

In particular, taking in (6) \( m = 0 \) we obtain for \( x > 0 \),

\[
\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x}
\] (8)
and taking in (7) \( m = 1 \) and \( n = 1 \), we obtain for \( x > 0 \)

\[
\frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} < \frac{1}{x} - \psi'(x + 1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \tag{9}
\]

The inequalities (8) and (9) play an important role in the proof of the theorem 2 in Section 2.

§2. Inequalities for Euler’s Constant

Euler’s constant \( \gamma = 0.57721 \cdots \) is defined by

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right)
\]

It is of interest to investigate the bounds for the expression \( \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma \). The inequality

\[
\frac{1}{2n} - \frac{1}{8n^2} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n}
\]

is called in literature Franel’s inequality [3, Ex. 18].

It is given in [1, p. 258] that \( \psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma \), and then we have get

\[
\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma = \psi(n + 1) - \ln n. \tag{10}
\]

Taking in (6) \( x = n \) we obtain that

\[
\frac{1}{2n} - \sum_{j=1}^{2n} \frac{B_{2j}}{2j} \frac{1}{n^{2j}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n} - \sum_{j=1}^{2n} \frac{B_{2j}}{2j} \frac{1}{n^{2j}}. \tag{11}
\]

The inequality (11) provides closer bounds for \( \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma \).

L.Tóth [5, p. 264] proposed the following problems:

(i) Prove that for every positive integer \( n \) we have

\[
\frac{1}{2n + \frac{2}{3}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}.
\]

(ii) Show that \( \frac{2}{5} \) can be replaced by a slightly smaller number, but that \( \frac{1}{3} \) can not be replaced by a slightly larger number.

The following Theorem 2 answers the problem due to L.Tóth.

**Theorem 2.** For every positive integer \( n \),

\[
\frac{1}{2n + a} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + b}, \tag{12}
\]

with the best possible constants
\[ a = \frac{1}{1-\gamma} - 2 \quad \text{and} \quad b = \frac{1}{3} \]

**Proof.** By (10), the inequality (12) can be rearranged as

\[ b < \frac{1}{\psi(n+1) - \ln n - 2n} \leq a. \]

Define for \( x > 0 \)
\[ \phi(x) = \frac{1}{\psi(x+1) - \ln x - 2x}. \]

Differentiating \( \phi \) and utilizing (8) and (9) reveals that for \( x > \frac{12}{5} \)

\[
\left( \psi(x+1) - \ln x \right)^2 \phi'(x) = \frac{1}{x} - \psi'(x+1) - 2(\psi(x+1) - \ln x)^2 \\
< \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left( \frac{1}{2x} - \frac{1}{12x^2} \right)^2 = \frac{12 - 5x}{360x^5} < 0,
\]

and then the function \( \phi \) strictly decreases with \( x > \frac{12}{5} \).

Straightforward calculation produces
\[
\phi(1) = \frac{1}{1-\gamma} - 2 = 0.36527211862544155\ldots,
\]
\[
\phi(2) = \frac{1}{\frac{1}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752\ldots,
\]
\[
\phi(3) = \frac{1}{\frac{1}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115\ldots.
\]

Therefore, the sequence
\[
\phi(n) = \frac{1}{\psi(n+1) - \ln n - 2n}, \quad n \in N
\]
is strictly decreasing. This leads to

\[
\lim_{n \to \infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1-\gamma} - 2.
\]

Making use of asymptotic formula of \( \psi \) (see [1, p. 259])
\[
\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \quad (x \to \infty),
\]

we conclude that
\[
\lim_{n \to \infty} \phi(n) = \lim_{x \to \infty} \phi(x) = \lim_{x \to \infty} \frac{\frac{1}{3} + O(x^{-2})}{1 + O(x^{-1})} = \frac{1}{3}.
\]

This completes the proof of the theorem 2.
References


