Some interesting properties of the Smarandache function

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Abstract The main purpose of this paper is using the elementary method to study the property of the Smarandache function, and give an interesting result.

Keywords Smarandache function; Additive property; Greatest prime divisor.

§1. Introduction and results

Let \( n \) be a positive integer, the famous Smarandache function \( S(n) \) is defined as following:

\[
S(n) = \min\{m : m \in \mathbb{N}, n|m!\}.
\]

About this function and many other Smarandache type function, many scholars have studied its properties, see [1], [2], [3] and [4]. Let \( p(n) \) denotes the greatest prime divisor of \( n \), it is clear that \( S(n) \geq p(n) \). In fact, \( S(n) = p(n) \) for almost all \( n \), as noted by Erdős [5]. This means that the number of \( n \leq x \) for which \( S(n) \neq p(n) \), denoted by \( N(x) \), is \( o(x) \). It is easily to show that \( S(p) = p \) and \( S(n) < n \) except for the case \( n = 4, n = p \). So there have a closely relationship between \( S(n) \) and \( \pi(x) \):

\[
\pi(x) = -1 + \sum_{n=2}^{[x]} \left\lfloor \frac{S(n)}{n} \right\rfloor,
\]

where \( \pi(x) \) denotes the number of primes up to \( x \), and \([x]\) denotes the greatest integer less than or equal to \( x \). For two integer \( m \) and \( n \), can you say \( S(mn) = S(m) + S(n) \) is true or false? It is difficult to say. For some \( m \) an \( n \), it is true, but for some other numbers it is false.

About this problem, J.Sandor [7] proved an very important conclusion. That is, for any positive integer \( k \) and any positive integers \( m_1, m_2, \cdots, m_k \), we have the inequality

\[
S\left( \prod_{i=1}^{k} m_i \right) \leq \sum_{i=1}^{k} S(m_i).
\]

This paper as a note of [7], we shall prove the following two conclusions:

**Theorem 1.** For any integer \( k \geq 2 \) and positive integers \( m_1, m_2, \cdots, m_k \), we have the inequality

\[
S\left( \prod_{i=1}^{k} m_i \right) \leq \prod_{i=1}^{k} S(m_i).
\]
Theorem 2. For any integer \( k \geq 2 \), we can find infinite group numbers \( m_1, m_2, \cdots, m_k \) such that:

\[
S \left( \prod_{i=1}^{k} m_i \right) = \sum_{i=1}^{k} S(m_i).
\]

§2. Proof of the theorems

In this section, we will complete the proof of the Theorems. First we prove a special case of Theorem 1. That is, for any positive integers \( m \) and \( n \), we have

\[
S(m)S(n) \geq S(mn).
\]

If \( m = 1 \) (or \( n = 1 \)), then it is clear that \( S(m)S(n) \geq S(mn) \). Now we suppose \( m \geq 2 \) and \( n \geq 2 \), so that \( S(m) \geq 2 \), \( S(n) \geq 2 \), \( mn \geq m + n \) and \( S(m)S(n) \geq S(m) + S(n) \). Note that \( m \mid S(m)!, \ n \mid S(n)! \), we have \( mn \mid S(m)!S(n)! \). Because \( S(m)S(n) \geq S(m) + S(n) \), we have \( (S(m) + S(n))! \mid (S(m)S(n))! \). That is, \( mn \mid S(m)!S(n)! \). From the definition of \( S(n) \) we may immediately deduce that

\[
S(mn) \leq S(m)S(n).
\]

Now the theorem 1 follows from \( S(mn) \leq S(m)S(n) \) and the mathematical induction.

Proof of Theorem 2. For any integer \( n \) and prime \( p \), if \( p^n \parallel n! \), then we have

\[
\alpha = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.
\]

Let \( n_i \) are positive integers such that \( n_i \neq n_j \), if \( i \neq j \), where \( 1 \leq i, j \leq k \), \( k \geq 2 \) is any positive integer. Since

\[
\sum_{r=1}^{\infty} \left\lfloor \frac{p^{n_i}}{p^r} \right\rfloor = p^{n_i-1} + p^{n_i-2} + \cdots + 1 = \frac{p^{n_i} - 1}{p - 1}.
\]

For convenient, we let \( u_i = \frac{p^{n_i} - 1}{p - 1} \). So we have

\[
S(p^{u_i}) = p^{n_i}, \quad i = 1, 2, \cdots, k.
\]

In general, we also have

\[
\sum_{r=1}^{\infty} \left[ \frac{k}{p^r} \right] p^{n_i} = \frac{k}{p - 1} = \sum_{i=1}^{k} u_i.
\]

So

\[
S \left( p^{u_1 + u_2 + \cdots + u_k} \right) = \sum_{i=1}^{k} p^{n_i}.
\]
Combining (1) and (2) we may immediately obtain

\[ S \left( \prod_{i=1}^{k} p^{m_i} \right) = \sum_{i=1}^{k} S(p^{m_i}). \]

Let \( m_i = p^{n_i} \), noting that there are infinity primes \( p \) and \( n_i \), we can easily get Theorem 2.

This completes the proof of the theorems.

References


