On Additive Analogues of Certain Arithmetic Functions

József Sándor
Department of Mathematics,
Babeș-Bolyai University, 3400 Cluj-Napoca, Romania

1. The Smarandache, Pseudo-Smarandache, resp. Smarandache-simple functions are defined as ([7], [6])

\[ S(n) = \min \{ m \in \mathbb{N} : n|m! \}, \quad (1) \]
\[ Z(n) = \min \left\{ m \in \mathbb{N} : n\frac{m(m+1)}{2} \right\}, \quad (2) \]
\[ S_p(n) = \min \{ m \in \mathbb{N} : p^n|m! \} \text{ for fixed primes } p. \quad (3) \]

The duals of $S$ and $Z$ have been studied e.g. in [2], [5], [6]:

\[ S_*(n) = \max \{ m \in \mathbb{N} : m!|n \}, \quad (4) \]
\[ Z_*(n) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2}|n \right\}. \quad (5) \]

We note here that the dual of the Smarandache simple function can be defined in a similar manner, namely by

\[ S_{p^*}(n) = \max \{ m \in \mathbb{N} : m!|p^n \} \quad (6) \]

This dual will be studied in a separate paper (in preparation).
2. The additive analogues of the functions $S$ and $S_*$ are real variable functions, and have been defined and studied in paper [3]. (See also our book [6], pp. 171-174). These functions have been recently further extended, by the use of Euler’s gamma function, in place of the factorial (see [1]). We note that in what follows, we could define also the additive analogues functions by the use of Euler’s gamma function. However, we shall apply the more transparent notation of a factorial of a positive integer.

The additive analogues of $S$ and $S_*$ from (1) and (4) have been introduced in [3] as follows:

$$S(x) = \min \{m \in \mathbb{N} : x \leq m!\}, \quad S : (1, \infty) \to \mathbb{R},$$

(resp. 

$$S_*(x) = \max \{m \in \mathbb{N} : m! \leq x\}, \quad S_* : [1, \infty) \to \mathbb{R}$$)

Besides of properties relating to continuity, differentiability, or Riemann integrability of these functions, we have proved the following results:

**Theorem 1.**

$$S_*(x) \sim \frac{\log x}{\log \log x} \quad (x \to \infty)$$

*(the same for $S(x)$).*

**Theorem 2.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$ (the same for $S_*(n)$ replaced by $S(n)$).

3. The additive analogues of $Z$ and $Z_*$ from (2), resp. (4) will be defined as

$$Z(x) = \min \left\{ m \in \mathbb{N} : x \leq \frac{m(m+1)}{2} \right\},$$

$$Z_*(x) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \leq x \right\}$$
In (11) we will assume \( x \in (0, +\infty) \), while in (12) \( x \in [1, +\infty) \).

The two additive variants of \( S_p(n) \) of (3) will be defined as

\[
P(x) = S_p(x) = \min \{ m \in \mathbb{N} : \ p^x \leq m! \};
\]

(13)

(where in this case \( p > 1 \) is an arbitrary fixed real number)

\[
P_*(x) = S_{p*}(x) = \max \{ m \in \mathbb{N} : \ m! \leq p^x \};
\]

(14)

From the definitions follow at once that

\[
Z(x) = k \iff x \in \left( \frac{(k - 1)k}{2}, \frac{k(k + 1)}{2} \right) \text{ for } k \geq 1
\]

(15)

\[
Z_*(x) = k \iff x \in \left[ \frac{k(k + 1)}{2}, \frac{(k + 1)(k + 2)}{2} \right)
\]

(16)

For \( x \geq 1 \) it is immediate that

\[
Z_*(x) + 1 \geq Z(x) \geq Z_*(x)
\]

(17)

Therefore, it is sufficient to study the function \( Z_*(x) \).

The following theorems are easy consequences of the given definitions:

**Theorem 3.**

\[
Z_*(x) \sim \frac{1}{2} \sqrt{8x + 1} \quad (x \to \infty)
\]

(18)

**Theorem 4.**

\[
\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^\alpha} \quad \text{is convergent for } \alpha > 2
\]

(19)

and divergent for \( \alpha \leq 2 \). The series \( \sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^{\alpha}} \) is convergent for all \( \alpha > 0 \).

**Proof.** By (16) one can write

\[
\frac{k(k + 1)}{2} \leq x < \frac{(k + 1)(k + 2)}{2}, \text{ so } k^2 + k - 2x \leq 0
\]

and \( k^2 + 3k + 2 - 2x > 0 \). Since the solutions of these quadratic equations are \( k_{1,2} = \frac{-1 \pm \sqrt{8x + 1}}{2} \), resp. \( k_{3,4} = \frac{-3 \pm \sqrt{8x + 1}}{2} \), and remarking that \( \frac{\sqrt{8x + 1} - 3}{2} \geq \)}
1 \Rightarrow x \geq 3$, we obtain that the solution of the above system of inequalities is:

\[
\begin{cases}
  k \in \left[1, \frac{\sqrt{1+8x} - 1}{2}\right] & \text{if } x \in \left[1, 3\right); \\
  k \in \left(\frac{\sqrt{1+8x} - 3}{2}, \frac{\sqrt{1+8x} - 1}{2}\right] & \text{if } x \in \left[3, +\infty\right).
\end{cases}
\] (20)

So, for $x \geq 3$

\[
\frac{\sqrt{1+8x} - 3}{2} < Z_*(x) \leq \frac{\sqrt{1+8x} - 1}{2}
\] (21)

implying relation (18).

Theorem 4 now follows by (18) and the known fact that the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^\theta}$ is convergent only for $\theta > 1$.

The things are slightly more complicated in the case of functions $P$ and $P_*$. Here it is sufficient to consider $P_*$, too.

First remark that

\[
P_*(x) = m \Leftrightarrow x \in \left[\frac{\log m!}{\log p}, \frac{\log(m+1)!}{\log p}\right].
\] (22)

The following asymptotic results have been proved in [3] (Lemma 2) (see also [6], p. 172)

\[
\log m! \sim m \log m, \quad \frac{m \log m!}{\log m!} \sim 1, \quad \frac{\log m!}{\log(m+1)!} \sim 1 \quad (m \to \infty)
\] (23)

By (22) one can write

\[
\frac{m \log m!}{\log m!} - \frac{m}{\log m!} \log \log p \leq \frac{m \log x}{\log m!} \leq \frac{m \log m!}{\log m!} - (\log \log p) \frac{m}{\log m!},
\]

giving $\frac{m \log x}{\log m!} \to 1 \quad (m \to \infty)$, and by (23) one gets $\log x \sim \log m$. This means that:

**Theorem 5.**

\[
\log P_*(x) \sim \log x \quad (x \to \infty)
\] (24)

The following theorem is a consequence of (24), and a convergence theorem established in [3]:
Theorem 6. The series \( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\log \log n}{\log P_r(n)} \right)^\alpha \) is convergent for \( \alpha > 1 \) and divergent for \( \alpha \leq 1 \).

Indeed, by (24) it is sufficient to study the series \( \sum_{n \geq n_0} \frac{1}{n} \left( \frac{\log \log n}{\log n} \right)^\alpha \) (where \( n_0 \in \mathbb{N} \) is a fixed positive integer). This series has been proved to be convergent for \( \alpha > 1 \) and divergent for \( \alpha \leq 1 \) (see [6], p. 174).

References


