SOME ELEMENTARY ALGEBRAIC CONSIDERATIONS INSPIRED BY THE SMARANDACHE FUNCTION

by

E.Radescu, N.Radescu, C.Dumitrescu

It is known that the Smarandache function $S:N^* \dashrightarrow N^*$, $S(n)=\min\{k|n \text{ divides } k!\}$ satisfies

(i) S is surjective

(ii) $S([m,n]) = \max\{S(m),S(n)\}$, where $[m,n]$ is the smallest common multiple of m and n.

That is on N^* there are considered both of the divisibility order $\leq_d$ ($m \leq_d n$ if and only if m divide n) and the usual order $\leq$. Of course the algebraic usual operations "+" and "$\cdot$" play also an important role in the description of the properties of S. For instance it is said that [1]:

$$\max\{S(k^k),S(n^n)\} \leq S((kn)^{kn}) \leq nS(k^k)+kS(n^n).$$

If we consider the universal algebra (N^*,Ω), with $\Omega=\{V_d,\Phi_0\}$, where $V_d:(N^*)^2 \dashrightarrow N^*$ is given by, $m\ V_d\ n=[m,n]$, and $\Phi_0:(N^*)^0 \dashrightarrow N^*$, is given by $\Phi_0(\{\Phi\})=1=e_{V_d}$ and analogously the universal algebra (N^*,Ω') with $\Omega'=\{V,\Psi_0\}$, where $V:(N^*)^2 \dashrightarrow N^*$, is defined by $m\vee n=\max\{m,n\}$, and $\Psi_0:(\mathbf{N}^*)^0\dashrightarrow\mathbf{N}^*$ is defined by $\Psi_0(\{\Phi\})=1=e_\vee$, then it results:

1. PROPOSITION. Let $\overline{\mathbf{N}}=\{S^-(k)\mid k\in\mathbf{N}^*\}$, where $S^-(k)=\{x\in\mathbf{N}^*\mid S(x)=k\}$. Then

(a) $\overline{\mathbf{N}}$ is countable ($card\mathbf{N}^*=alef\ zero$).

(b) on $\overline{\mathbf{N}}$ may be defined an universal algebra, isomorfe with $(\mathbf{N}^*,\Omega')$.

Proof. (b) Let $\omega:(\overline{\mathbf{N}})^2\dashrightarrow\overline{\mathbf{N}}$ be defined by $\omega(S^-(a),S^-(b))=S^-(c)$, where $C=S(x\vee_d y)$, with $x\in S^-(a)$, $y\in S^-(b)$.

Then ω is well defined because if $x_1\in S^-(a)$, $y_1\in S^-(b)$ the

$S(x_1\vee_d y_1)=S(x_1)\vee S(y_1)=a\vee b=S(x)\vee S(y)=S(x\vee_d y)=c$.

Example. $\omega(S^-(23),S^-(14))=S^-(23)$ because if for instance $x=46\in S^-(23)$ and $y=49\in S^-(14)$ then $46\,V_d49=2254$ and $S(2254)=23$.

In fact, because $c=S(x\vee_d y)=S(x)\vee S(y)=a\vee b$, it results that ω is defined by

$$\omega(S^-(a),S^-(b))=S^-(a\vee b).$$

We define now $\omega_0:(\overline{\mathbf{N}})^0\dashrightarrow\overline{\mathbf{N}}$ by $\omega_0(\{\Phi\})=S^-(1)$.

Let us note $S^-(1)=e_\omega$. Then

$$\forall S^-(k)\in\overline{\mathbf{N}}\quad \omega(S^-(k),e_\omega)=\omega(e_\omega,S^-(k))=S^-(k).$$

Then $(\overline{\mathbf{N}},\overline{\Omega})$ is an universal algebra if $\overline{\Omega}=\{\omega,\omega_0\}$.

It may be defined $h:\overline{\mathbf{N}}\dashrightarrow\mathbf{N}^*$ an isomorphism between $(\overline{\mathbf{N}},\overline{\Omega})$ and $(\mathbf{N}^*,\Omega')$, by $h(S^-(k))=k$.

We have

$$\forall S^{-}(a), S^{-}(b) \in \overline{\mathbf{N}} \quad h(\omega(S^{-}(a), S^{-}(b))) = h(S^{-}(a \vee b) =$$

$$= a \vee b = h(S^{-}(a)) \vee h(S^{-}(b))$$

that is h is a morphism.

Of course $h(\omega_0(\{\Phi\})) = \Psi_0(\{\Phi\})$ and **h** is injective.

Indeed, $h(S^{-}(a)) = h(S^{-}(b)) \Leftrightarrow a = b$ and then

$x \in S^{-}(a) \Leftrightarrow S(x) = a = b \Rightarrow x \in S^{-}(b) \Rightarrow S^{-}(a) \subset S^{-}(b)$ and analogously

$S^{-}(b) \subset S^{-}(a)$, so $S^{-}(a) = S^{-}(b)$.

From the surjectivity of S it results that **h** is surjective, because for every $k \in \mathbf{N}^{*}$ it exists $x \in \mathbf{N}^{*}$ such that $S(x) = k$, so $S^{-}(k) \neq \Phi$ and $h(S^{-}(k)) = k$.

Then we have $(\overline{\mathbf{N}}, \overline{\Omega}) \simeq (\mathbf{N}^{*}, \Omega')$ and from the bijectivity of **h** it results $card\,\overline{\mathbf{N}} = card\,\mathbf{N}^{*}$, that is the assertion (a).

Remarks (i) An other proof of Proposition 1 may be made as follows;

Let ρ_s be the equivalence associated with the function S

$$x \rho_s y \Leftrightarrow S(x) = S(y).$$

Because S is a morphism between $(\mathbf{N}^{*}, \Omega)$ and $(\mathbf{N}^{*}, \Omega')$ it results that ρ_s is a congruence and so we can define on $\dfrac{\mathbf{N}^{*}}{\rho_s}$ the operations

ω and ω_0 by

$\omega : (\mathbf{N}^*/\rho_S)^2 \dashrightarrow \mathbf{N}^*/\rho_S,\quad \omega(\hat{x},\hat{y}) = xV_d y;$

$\omega_0 : (\mathbf{N}^*/\rho_S)^2 \dashrightarrow \mathbf{N}^*/\rho_S,\quad \omega_0(\{\Phi\}) = \hat{1}.$

Moreover, $\mathbf{N}^*/\rho_S = \overline{\mathbf{N}}$ and so it is constructed the universal algebra $(\overline{\mathbf{N}}, \overline{\Omega})$, with $\overline{\Omega} = \{\omega, \omega_0\}$. That because $S: (\mathbf{N}^*, \Omega) \dashrightarrow (\mathbf{N}^*, \Omega')$ is a morphism so by a well known isomorphism theorem it results that $(\mathbf{N}^*/\rho_S) \simeq ImS$ so $(\overline{\mathbf{N}}, \overline{\Omega}) \simeq (\mathbf{N}^*, \Omega')$. That is we have a proof for (b), the morphism being $\alpha : \overline{\mathbf{N}} \dashrightarrow \mathbf{N}^*$, $\alpha(\hat{x}) = S(x)$.

(ii) Proposition 1 is an argument to consider the functions

$S^{-1}_{\min} : \mathbf{N}^* \dashrightarrow \mathbf{N}^*,\quad S^{-1}_{\min}(k) = \min S^-(k)$

$S^{-1}_{\max} : \mathbf{N}^* \dashrightarrow \mathbf{N}^*,\quad S^{-1}_{\max}(k) = \max S^-(k)$ (sec [4])

whose properties we shall present in a future note.

(iii) The graph

$$G = \{(x,y) \in \mathbf{N}^* X \mathbf{N}^* \ / \ y = S(x)\}$$

is a subalgebra of the universal algebra $(\mathbf{N}^* X \mathbf{N}^*, \Omega)$, where $\Omega = \{\omega, \omega_0\}$, with $\omega : (\mathbf{N}^* X \mathbf{N}^*)^2 \dashrightarrow \mathbf{N}^* X \mathbf{N}^*$, defined by $\omega((x_1,y_1),(x_2,y_2)) = (x_1 V d x_2, y_1 V y_2)$ and $\omega_0 : (\mathbf{N}^* x \mathbf{N}^*)^0 \dashrightarrow \mathbf{N}^* x \mathbf{N}^*$, defined by $\omega_0(\{\Phi\}) = (\phi_0(\{\Phi\}), \Psi_0(\{\Phi\})) = (1,1)$.

Indeed G is a subalgebra of the universal algebra $(\mathbf{N}^* x \mathbf{N}^*, \Omega)$ if for every $(x_1, y_1), (x_2, y_2) \in G$ it results $\omega((x_1, y_1), (x_2, y_2) \in G$ and $\omega_0(\{\Phi\}) \in G$. But

$\omega((x_1,y_1),(x_2,y_2)) = (x_1 V_d x_2, y_1 V y_2) = (x_1 V_d x_2, S(x_1) V S(x_2)) = (x_1 V_d x_2.$

and $\omega_0(\{\Phi\}) \in G$ if and only if $(1,1) \in G$.

That is $(1, S(1)) \in G$.

In fact the algebraic property is more complete in the sense that $f: A \dashrightarrow B$ is a morphism between the universal algebras (A, Ω) and (B, Ω) of the some kind τ if and only if the graph F of the functional relation f is a subalgebra of the universal algebra $(A x B, \Omega)$.

Then the importance of remark (iii) consist in the fact that it is possible to underline some properties of the Smarandache function starting from the above mentioned subalgebra of the universal algebra $(\mathbf{N}^* x \mathbf{N}^*, \Omega)$.

References

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Current Address: University of Craiova, Dept of Math. Craiova (1100), România.