SOME ELEMENTARY ALGEBRAIC CONSIDERATIONS
INSPIRED BY THE SMARANDACHE FUNCTION
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It is known that the Smarandache function \( S: \mathbb{N} \to \mathbb{N} \),
\( S(n) = \min\{ k | n \text{ divides } k! \} \) satisfies

(i) \( S \) is surjective

(ii) \( S([m,n]) = \max \{ S(m), S(n) \} \), where \([m,n]\) is the smallest
common multiple of \( m \) and \( n \).

That is on \( \mathbb{N} \) there are considered both of the divisibility
order \( \preceq_d \) ( \( m \preceq_d n \) if and only if \( m \text{ divide } n \) ) and the usual order \( \preceq \).
Of course the algebraic usual operations "+" and "·" play also an
important role in the description of the properties of \( S \).
For instance it is said that [1]:

\[
\max \{ S(k^t), S(n^t) \} \leq S((kn)^t) \leq nS(k^t) + kS(n^t).
\]

If we consider the universal algebra \((\mathbb{N}', \Omega)\), with
\( \Omega = \{ V_d, \phi_0 \} \), where \( V_d: (\mathbb{N}')^2 \to \mathbb{N}' \) is given by, \( m V_d n = [m,n] \), and
\( \phi_0: (\mathbb{N}')^2 \to \mathbb{N}' \), is given by \( \phi_0((\phi)) = 1 = \phi V_d \) and analogously the
universal algebra \((\mathbb{N}', \Omega')\) with \( \Omega' = \{ V, \Phi \} \), where \( V: (\mathbb{N}')^2 \to \mathbb{N}' \), is
defined by \( m \lor n = \max(m, n) \), and \( \Psi_0 : (N^*)^\circ \longrightarrow N^* \) is defined by

\[ \Psi_0(\emptyset) = 1 = e_\emptyset, \]

then it results:

1. **PROPOSITION.** Let \( \overline{N} = \{ S^-(k) \mid k \in N^* \} \), where \( S^-(k) = \{ x \in N^* \mid S(x) = k \} \). Then

(a) \( \overline{N} \) is countable (\( \text{card} N^* = \aleph_0 \)).

(b) on \( \overline{N} \) may be defined an universal algebra, isomorfe with \( (N^*, \mathcal{Q}', \cdot, +) \).

**Proof.** (b) Let \( \omega : (\overline{N})^2 \longrightarrow \overline{N} \) be defined by

\[ \omega(S^-(a), S^-(b)) = S^-(c), \]

where \( C = S(x \lor y) \), with \( x \in S^-(a), y \in S^-(b) \).

Then \( \omega \) is well defined because if \( x_1 \in S^-(a), y_1 \in S^-(b) \) the

\[ S(x_1 \lor y_1) = S(x_1) \lor S(y_1) = a \lor b = S(x \lor y) = c. \]

Example. \( \omega(S^-(23), S^-(14)) = S^-(23) \) because if for instance \( x = 46 \in S^-(23) \) and \( y = 49 \in S^-(14) \) then \( 46 \lor 49 = 2254 \) and \( S(2254) = 23 \).

In fact, because \( C = S(x \lor y) = S(x) \lor S(y) = a \lor b \), it results that \( \omega \) is defined by

\[ \omega(S^-(a), S^-(b)) = S^-(a \lor b). \]

We define now \( \omega_0 : (\overline{N})^\circ \longrightarrow \overline{N} \) by \( \omega_0(\emptyset) = S^-(1) \).

Let us note \( S^-(1) = e^\emptyset \). Then

\[ \forall S^-(k) \in \overline{N} \omega(S^-(k), e^\emptyset) = \omega(e^\emptyset, S^-(k)) = S^-(k). \]

Then \( (\overline{N}, \mathcal{Q}) \) is an universal algebra if \( \mathcal{Q} = \{ \omega, \omega_0 \} \).

It may be defined \( h : \overline{N} \longrightarrow N^* \) an isomorphism between \( (\overline{N}, \mathcal{Q}) \) and \( (N^*, \mathcal{Q}', \cdot, +) \), by \( h(S^-(k)) = k \).
We have
\[ \forall S^-(a), S^-(b) \in \mathbb{N} \ h(\omega(S^-(a), S^-(b))) = h(S^-(a \lor b)) = a \lor b = h(S^-(a)) \lor h(S^-(b)) \]
that is \( h \) is a morphism.

Of course \( h(S^*(\emptyset)) = T_2(\emptyset) \) and \( h \) is injective.

Indeed, \( h(S^-(a)) = h(S^-(b)) \iff a = b \) and then

\[ x \in S^-(a) \iff S(x) = a = b = x \in S^-(b) \Rightarrow S^-(a) \subseteq S^-(b) \text{ and analogously} \]

\( S^-(b) \subseteq S^-(a), \text{ so } S^-(a) = S^-(b). \)

From the surjectivity of \( S \) it results that \( h \) is surjective, because for every \( k \in \mathbb{N}^* \) it exists \( x \in \mathbb{N}^* \) such that \( S(x) = k \), so \( S^-(k) \neq \emptyset \) and \( h(S^-(k)) = k \).

Then we have \( (\mathbb{N}, Q) \cong (\mathbb{N}^*, Q') \) and from the bijectivity of \( h \) it results \( \text{card} \mathbb{N} = \text{card} \mathbb{N}^* \), that is the assertion (a).

Remarks (i) An other proof of Proposition 1 may be made as follows:

Let \( \rho_s \) be the equivalence associated with the function \( S \)

\[ x \rho_s y \iff S(x) = S(y). \]

Because \( S \) is a morphism between \( (\mathbb{N}^*, \Omega) \) and \( (\mathbb{N}^*, \Omega') \) it results that \( \rho_s \) is a congruence and so we can define on \( \mathbb{N}^* \) the operations \( \omega \) and \( \omega_s \) by
\[ \omega : (N^*/\rho_s)^2 \longrightarrow N^*/\rho_s, \quad \omega (x, y) = x \lor y; \]

\[ \omega_0 : (N^*/\rho_s)^2 \longrightarrow N^*/\rho_s, \quad \omega_0 (\emptyset) = 1. \]

Moreover, \( N^*/\rho_s = \overline{N} \) and so it is constructed the universal algebra \((\overline{N}, \overline{\Omega})\), with \( \overline{\Omega} = (\omega, \omega) \). That because \( S : (N^*, \Omega) \longrightarrow (N^*, \Omega') \) is a morphism so by a well known isomorphism theorem it results that \( (N^*/\rho_s) \cong \text{Im} S \) so \((\overline{N}, \overline{\Omega}) \cong (N^*, \Omega')\). That is we have a proof for (b), the morphism being \( \alpha : \overline{N} \longrightarrow N^*, \alpha (x) = S(x) \).

(ii) Proposition 1 is an argument to consider the functions

\[ S^{-1}_{\text{min}} : N^* \longrightarrow N^*, \quad S^{-1}_{\text{min}} (k) = \min S^{-1} (k) \]

\[ S^{-1}_{\text{max}} : N^* \longrightarrow N^*, \quad S^{-1}_{\text{max}} (k) = \max S^{-1} (k) \quad (\text{sec } [4]) \]

whose properties we shall present in a future note.

(iii) The graph

\[ G = \{ (x, y) \in N^* \times N^* / y = S(x) \} \]

is a subalgebra of the universal algebra \((N^* \times N^*, \Omega)\), where \( \Omega = (\omega, \omega) \), with \( \omega : (N^* \times N^*)^2 \longrightarrow N^* \times N^* \), defined by

\[ \omega ((x_1, y_1), (x_2, y_2)) = (x_1 \lor x_2, y_1 \lor y_2) \]

and \( \omega_0 : (N^* \times N^*)^6 \longrightarrow N^* \times N^* \), defined by \( \omega_0 (\emptyset) = (\phi_0 (\emptyset), \psi_0 (\emptyset)) = (1, 1) \).

Indeed \( G \) is a subalgebra of the universal algebra \((N^* \times N^*, \Omega)\) if for every \( (x_1, y_1), (x_2, y_2) \in G \) it results \( \omega ((x_1, y_1), (x_2, y_2) \in G \) and \( \omega_0 (\emptyset) \in G \). But

\[ \omega ((x_1, y_1), (x_2, y_2)) = (x_1 \lor x_2, y_1 \lor y_2) = (x_1 \lor x_2, S(x_2) \lor S(x_2)) = (x_2 \lor x_2) \]

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and $\omega_0(\emptyset) \in G$ if and only if $(1, 1) \in G$.

That is $(1, S(1)) \in G$.

In fact the algebraic property is more complete in the sense that $f: A \rightarrow B$ is a morphism between the universal algebras $(A, \Omega)$ and $(B, \Omega)$ of the same kind if and only if the graph $P$ of the functional relation $f$ is a subalgebra of the universal algebra $(A \times B, \Omega)$.

Then the importance of remark (iii) consists in the fact that it is possible to underline some properties of the Smarandache function starting from the above mentioned subalgebra of the universal algebra $(N^* \times N^*, \Omega)$.

References


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