On The Irrationality Of Certain Alternative Smarandache Series

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1. Let \( S(n) \) be the Smarandache function. In paper [1] it is proved the irrationality of \( \sum_{n=1}^{\infty} \frac{S(n)}{n!} \). We note here that this result is contained in the following more general theorem (see e.g. [2]).

**Theorem 1** Let \( (x_n) \) be a sequence of natural numbers with the properties: (1) there exists \( n_0 \in \mathbb{N}^* \) such that \( x_n \leq n \) for all \( n \geq n_0 \); (2) \( x_n < n-1 \) for an infinity of \( n \); (3) \( x_m > 0 \) for infinitely many \( m \). Then the series \( \sum_{n=1}^{\infty} \frac{x_n}{n!} \) is irrational.

By letting \( x_n = S(n) \), it is well known that \( S(n) \leq n \) for \( n \geq n_0 = 1 \), and \( S(n) \leq \frac{2}{3}n \) for \( n > 4 \), composite. Clearly, \( \frac{2}{3}n < n-1 \) for \( n > 3 \). Thus the irrationality of the second constant of Smarandache ([1]) is contained in the above result.

2. We now prove a result on the irrationality of the alternating series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} S(n)}{n!} \).

We can formulate our result more generally, as follows:

**Theorem 2** Let \( (a_n), (b_n) \) be two sequences of positive integers having the following properties: (1) \( n|a_1a_2\ldots a_n \) for all \( n \geq n_0 \) \( (n_0 \in \mathbb{N}^*) \); (2) \( \frac{b_{n+1}}{a_{n+1}} < b_n \leq a_n \) for \( n \geq n_0 \); (3) \( b_m < a_m \), where \( m \geq n_0 \) is composite. Then the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_n}{a_1a_2\ldots a_n} \) is convergent and has an irrational value.

**Proof:** It is sufficient to consider the series \( \sum_{n=a_1}^{\infty} \frac{(-1)^{n-1} b_n}{a_1a_2\ldots a_n} \). The proof is very similar (in some aspect) to Theorem 2 in our paper [3]. Let \( x_n = \frac{b_n}{a_1a_2\ldots a_n} \) \( (n \geq n_0) \).
Then $x_n \leq \frac{1}{a_1 \cdots a_{n-1}} \to 0$ since (1) gives $a_1 \cdots a_k \geq k \to \infty$ (as $k \to \infty$). On the other hand, $x_{n+1} < x_n$ by the first part of (2). Thus Leibnitz criteria assures the convergence of the series. Let us now assume, on the contrary, that the series has a rational value, say $\frac{a}{k}$. First we note that we can choose $k$ in such a manner that $k+1$ is composite, and $k > n_0$. Indeed, if $k+1 = 1$ (prime), then $\frac{a}{p-1} = \frac{ca}{c(p-1)}$. Let $c = 2ar^2 + 2r$, where $r$ is arbitrary. Then $2a(2ar^2 + 2r) + 1 = (2ar + 1)^2$, which is composite. Since $r$ is arbitrary, we can assume $k > n_0$. By multiplying the sum with $a_1a_2 \cdots a_k$, we can write:

$$a = \frac{a_1 \cdots a_k}{k} = \sum_{n=0}^{k} (-1)^{n+1} \frac{a_1 \cdots a_k}{a_1 \cdots a_n} \cdot b_n + (-1)^k \left( \frac{b_{k+1}}{a_1} - \frac{b_{k+2}}{a_1a_{k+2}} + \cdots \right).$$

The alternating series on the right is convergent and must have an integer value. But it is well known its value lies between $\frac{b_{k+1}}{a_1} - \frac{b_{k+2}}{a_1a_{k+2}}$ and $\frac{b_{k+1}}{a_1}$. Here $\frac{b_{k+1}}{a_1} - \frac{b_{k+2}}{a_1a_{k+2}} > 0$ on base of (3). On the other hand $\frac{b_{k+1}}{a_1} < 1$, since $k+1$ is a composite number. Since an integer number has a value between 0 and 1, we have obtained a contradiction, finishing the proof of the theorem.

**Corollary** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{S(n)}{n!}$ is irrational.

**Proof:** Let $a_n = n$. Then condition (1) of Theorem 2 is obvious for all $n$, (2) is valid with $n_0 = 2$, since $S(n) \leq n$ and $S(n+1) \leq n+1 = (n+1) \cdot 1 < (n+1)S(n)$ for $n \geq 2$.

For composite $m$ we have $S(m) \leq \frac{2}{3} m < m$, thus condition (3) is verified, too.

**References:**

