ON THE BALU NUMBERS

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Abstract. In this paper we prove that there are only finitely many Balu numbers.

Key words. Smarandache factor partition, number of divisors, Balu number, finiteness.

1. Introduction

For any positive integer \( n \), let \( d(n) \) and \( f(n) \) be the number of distinct divisors and the Smarandache factor partitions of \( n \) respectively. If \( n \) is the least positive integer satisfying

\[
d(n) = f(n) = r
\]

for some fixed positive integers \( r \), then \( n \) is called a Balu number. For example, \( n=1, 16, 36 \) are Balu numbers. In [4], Murthy proposed the following conjecture.

Conjecture. There are finitely many Balu numbers.

In this paper we completely solve the mentioned question. We prove the following result.

Theorem. There are finitely many Balu numbers.

2. Preliminaries

For any positive integer \( n \) with \( n>1 \), let

\[
a_1 \ a_2 \ a_k
\]

\( n = p_1 \ p_2 \ ... \ p_k \)

be the factorization of \( n \).

Lemma 1 ([1, Theorem 273]). \( d(n) = (a_1 + 1)(a_2 + 1)...(a_k + 1) \).
Lemma 2. Let $a, p$ be positive integers with $p>1$, and let
\[(3) \quad b = \left( \frac{1}{2} \sqrt{1+8a} - \frac{1}{2} \right).\]
Then $p^a$ can be written as a product of $b$ distinct positive integers
\[(4) \quad p^a = p_1 p_2 \ldots p^{b-1} p^{b-1}.
\]
Proof. We see from (3) that $a \geq 1+2+\ldots+(b-1)+b$. Thus, the lemma is true.

Lemma 3. For any positive integer $m$, let $Y(m)$ be the $m$-th Bell number. Then we have
\[(5) \quad f(n) \geq Y(c),\]
where
\[(6) \quad c=b_1 + b_2 \ldots + b_k\]
and
\[(7) \quad b_i = \left( \frac{1}{2} \sqrt{1+8a_i} - \frac{1}{2} \right), \quad i=1,2,\ldots,k.\]
Proof. Since $p_1, p_2, \ldots, p_k$ are distinct primes in the factorization (2) of $n$, by Lemma 2, we see from (6) and (7) that $n$ can be written as a product of $c$ distinct positive integers
\[(8) \quad n = \prod_{i=1}^{k} p_i \left( \frac{a_r - b_i (b_i - 1)/2}{b_r} \right) \prod_{j=1}^{b_r} p_i.\]
Therefore, by (6) and (8), we get
\[(9) \quad f(n) \geq F(1\#c),\]
where $F(1\#c)$ is the number of Smarandache factor partitions of a product of $c$ distinct primes. Further, by [2, Theorem], we have
\[(10) \quad F(1\#c) = Y(c).\]
Thus, by (9) and (10), we obtain (5). The lemma is
proved.

Lemma 4 ([3]). \( \log Y(m) \sim m \log m \).

3. Proof of Theorem

We now suppose that there exist infinitely many Balu numbers. Let \( n \) be a Balu number, and let (2) be the factorization of \( n \). Further, let

(11) \( a = a_1 + a_2 + \ldots + a_k \).

Clear, if \( n \) is enough large, then \( a \) tends to infinite. Moreover, since

(12) \( b_i \geq \sqrt{a} \quad i = 1, 2, \ldots, k \),

by (7), we see from (6) that \( c \) tends to infinite too. Therefore, by Lemmas 1, 3 and 4, we get from (1), (2), (6) and (12) that

\[
1 = \frac{\log d(n)}{\log f(n)} \leq \frac{\sum_{i=1}^{k} \log(a_i+1)}{\left( \sum_{i=1}^{k} \sqrt{a_i} \right) \left( \log \sum_{i=1}^{k} \sqrt{a_i} \right)}
\]

\[
\leq \frac{\sum_{i=1}^{k} \log(a_i+1)}{k} < 1,
\]

a contradiction. Thus, there are finitely many Balu numbers. The theorem is proved.
References


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