ON SOME DIOPHANTINE EQUATIONS

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Let $S(n)$ be defined as the smallest integer such that $(S(n))!$ is divisible by $n$ (Smarandache Function). We shall assume that $S : \mathbb{N}^* \rightarrow \mathbb{N}^*$, $S(1) = 1$. Our purpose is to study a variety of Diophantine equations involving the Smarandache function. We shall determine all solutions of the equations (1), (3) and (8).

(1) $x^{S(y)} = S(x)^x$
(2) $x^{S(y)} = S(y)^x$
(3) $x^{S(y)} + S(x) = S(x)^y + x$
(4) $x^{S(y)} + S(y) = S(y)^x + x$
(5) $S(x)^x + x^2 = x^{S(y)} + S(x)^y$
(6) $S(y)^x + x^2 = x^{S(y)} + S(y)^y$
(7) $S(x)^x + x^3 = x^{S(y)} + S(x)^y$
(8) $S(y)^x + x^3 = x^{S(y)} + S(y)^y$.

For example, let us solve equation (1): We observe that if $x = S(x)$, then (1) holds. But $x = S(x)$ if and only if $x \in \{1, 2, 3, 4, 5, 7, \ldots \} \cup \{x \in \mathbb{N}^* ; x \text{ prime} \} \cup \{1, 4 \}$. If $x \geq 6$ is not a prime integer, then $S(x) < x$. We can write $x = S(x) + t, t \in \mathbb{N}^*$, which implies that $S(x)^{S(y) + t} = (S(x) + t)^{S(y) + t}$. Thus we have $S(x)^t = (1 + \frac{t}{S(y) + t})^{S(x)}$.

Applying the well-known result $(1 + \frac{t}{n})^n < 3^t$, for $n, k \in \mathbb{N}^*$, we have $S(x)^t < 3^t$ which implies that $S(x) < 3$ and consequently $x < 3$. This contradicts our choice of $x$.

Thus, the solutions of (1) are $A_1 = \{ x \in \mathbb{N}^* ; x \text{ prime} \} \cup \{1, 4 \}$.

Let us denote by $A_k$ the set of all solutions of the equation (k).

To find $A_3$ for example, we see that $(S(n), n) \in A_3$ for $n \in \mathbb{N}^*$. Now suppose that $x = S(y)$. We can show that $(x, y)$ does not belong to $A_3$ as follows: $1 < S(y) < x \Rightarrow S(y) \geq 2$ and $x \geq 3$. On the other hand, $S(y)^x - x^{S(y)} > S(y)^x - x^2 = (S(y) - x)(S(y)^{x-1} + xS(y)^{x-2} + \ldots + x^{x-2}) \geq (S(y) - x)(S(y)^{x-1} + xS(y) + x^2) = S(y)^x - x^2$.

Thus, $A_3 = \{ (x, y) ; y = n, x = S(n), n \in \mathbb{N}^* \}$.

To find $A_3$, we see that $x = 1$ implies $S(x) = 1$ and (3) holds.

If $S(x) = x$, (3) also holds.

If $x \geq 6$ is not a prime number, then $x > S(x)$.

Write $x = S(x) + t, t \in \mathbb{N}^* = \{1, 2, 3, \ldots \}$. Combining this with (3) yields

$S(x)^{S(y) + t} + S(x) + t = (S(x) + t)^{S(y) + t} \Leftrightarrow S(x)^{t} + t = (1 + tS(x))^{S(y) + t} < 3^t$

which implies $S(x) < 3$. This contradicts our choice of $x$. 

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Thus \( A_3 = \{ x \in \mathbb{N}^* ; x = \text{prime} \} \cup \{ 1, 4 \} \).

Now, we suppose that the reader is able to find \( A_2, A_3, \ldots, A_s \).

We next determine all positive integers \( x \) such that \( x = \sum_{k^2 \leq x} k^2 \).

Write \( 1^2 + 2^2 + \ldots + s^2 = x \) \( \quad (1) \)
\( s^2 < x \)
\( (s + 1)^2 \geq x \) \( \quad (2) \)

(1) implies \( x = s(s+1)(2s+1)/6 \). Combining this with (2) and (3) we have
\( 6s^2 < s(s+1)(2s+1) \) and \( 6(s+1)^2 \geq s(s+1)(2s+1) \). This implies that \( s \in \{2, 3\} \).
\( s = 2 \Rightarrow x = 5 \) and \( s = 3 \Rightarrow x = 14 \).

Thus \( x \in \{5, 14\} \).

In a similar way we can solve the equation \( x = \sum_{k^2 \leq x} k^2 \).

We find \( x \in \{9, 36, 100\} \).

We next show that the set \( M_s = \{ n \in \mathbb{N}^* ; n = \sum_{k^2 \leq n} k^2, p \geq 2 \} \) has at least \( \lceil p/\ln2 \rceil - 2 \) elements.

Let \( m \in \mathbb{N}^* \) such that \( m - 1 < p/\ln2 \)
and \( p/\ln2 < m \)
\( \quad (4) \)
\( \quad (5) \)

Write (4) and (5) as:
\( 2 < e^{s/\ln2} \)
\( \quad (6) \)
\( e^{s/\ln2} < 2 \)
\( \quad (7) \)

Write \( x_k = (1 + 1/k)^k \), \( y_k = (1 + 1/k)^{k+1} \).

It is known that \( x_s < c < y_s \) for every \( s, t \in \mathbb{N}^* \).

Combining this with (6) and (7) we have
\( x_s^{s/\ln2} < e^{s/\ln2} < 2 < e^{(s+1)/(s+1)} < y_s^{(s+1)/(s+1)} \) for every \( s, t \in \mathbb{N}^* \).

We have \( 2 < y_t^{(t+1)/(t+1)} \leq ((t+1)/(t+1))^{(t+1)/(t+1)} \) if \( (t+1)/(t+1) \leq 1 \).

So, if \( t \leq m - 2 \) we have \( 2 < ((t+1)/(t+1))^{(t+1)/(t+1)} \equiv 2 t^t < (t+1)^t \Rightarrow (t+1)^t - t^t > t^t \) \( \quad (8) \).

Let \( A_p(s) \) denote the sum \( 1^p + 2^p + \ldots + s^p \).

Proposition 1. \((t+1)^p > A_p(t) \) for every \( t \leq m-2, t \in \mathbb{N}^* \).

Proof. Suppose that \( A_p(t) \geq (t+1)^p \Rightarrow A_p(t-1) > (t+1)^p - t^p > t^p \)
\( A_p(t-2) > t^p - (t-1)^p > (t-1)^p \) \( \Rightarrow \ldots \) \( \Rightarrow A_p(1) > 2^p \) which is not true.

It is obvious that \( A_p(t) > t^p \) if \( t \in \mathbb{N}^* \), \( 2 \leq t \leq m-2 \) which implies \( A_p(t) \in M_s \)
for every \( t \in \mathbb{N}^* \) and \( 2 \leq t \leq m-2 \).

Therefore \( \text{card } M_s > m-3 = (m-1) - 2 = \lceil p/\ln2 \rceil - 2 \).

REFERENCES:


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