THE FIRST CONSTANT OF SMARANDACHE

by

Ion Cojocaru and Sorin Cojocaru

In this note we prove that the series \( \sum_{n=2}^{\infty} \frac{1}{S(n)!} \) is convergent to a real number \( s \in (0.717, 1.253) \) that we call the first constant constant of Smarandache.

It appears as an open problem, in [1], the study of the nature of the series \( \sum_{n=2}^{\infty} \frac{1}{S(n)!} \). We can write it as it follows:

\[
\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{3!} + \cdots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \cdots =
\]

\[
= \sum_{n=2}^{\infty} \frac{a(n)}{n!}, \text{ where } a(n) \text{ is the number of the equation } S(x) = n, n \in \mathbb{N}, n \geq 2 \text{ solutions}.
\]

It results from the equality \( S(x) = n \) that \( x \) is a divisor of \( n! \), so \( a(n) \) is smaller than \( d(n)! \).

So, \( a(n) < d(n)! \). (1)

Lemma 1. We have the inequality:

\[ d(n) \leq n - 2, \text{ for each } n \in \mathbb{N}, n \geq 7. \] (2)

Proof. Be \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) with \( p_1, p_2, \ldots, p_k \) prime numbers, and \( a_i \geq 1 \) for each \( i \in \{ 1, 2, \ldots, k \} \). We consider the function \( f : [1, \infty) \to \mathbb{R}, f(x) = a^x - x - 2, a \geq 2, \text{ fixed}. \) It is derivable on \( [1, \infty) \) and \( f(x) = a^x \ln a - 1. \) Because \( a \geq 2, \text{ and } x \geq 1 \) it results that \( a^x \geq 2, \) so \( a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1, \text{ i.e.}, f(x) > 0 \) for each \( x \in [1, \infty) \) and \( a \geq 2, \text{ fixed}. \) But \( f(1) = a - 3. \) It results that for \( a \geq 3 \) we have \( f(x) \geq 0, \text{ that means } a^x \geq x + 2. \)

Particularly, for \( a = p_i, i \in \{ 1, 2, \ldots, k \}, \) we obtain \( p_i^{a_i} \geq a_i + 2 \) for each \( p_i \geq 3. \)

If \( n = 2^s, s \in \mathbb{N}^*, \) then \( d(n) = s + 1 < 2^s - 2 = n - 2 \) for \( s \geq 3. \)

So we can assume \( k \geq 2, \text{ i.e. } p_2 \geq 3. \) It results the inequalities:
\[ p_1^{a_1} \geq a_1 + 1 \]
\[ p_2^{a_1} \geq a_2 + 2 \]
\[ \ldots \]
\[ p_k^{a_k} \geq a_k + 2, \]
et equivalent with
\[ p_1^{a_1} \geq a_1 + 1, \ p_2^{a_1} - 1 \geq a_2 + 1, \ldots, \ p_k^{a_k} - 1 \geq a_k + 1. \]  \hspace{1cm} (3)

Multiplying, member with member, the inequalities (3) we obtain:
\[ p_1^{a_1} (p_2^{a_2} - 1) \cdots (p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_k + 1) = d(n). \]  \hspace{1cm} (4)

Considering the obvious inequality:
\[ n - 2 \geq p_1^{a_1} (p_2^{a_2} - 1) \cdots (p_k^{a_k} - 1) \]  \hspace{1cm} (5)
and using (4) it results that:
\[ n - 2 \geq d(n) \text{ for each } n \geq 7. \]

**Lemma 2.** \( d(n!) < (n - 2)! \) for each \( n \in \mathbb{N}, n \geq 7. \)  \hspace{1cm} (6)

**Proof.** We ration through induction after \( n. \) So, for \( n = 7, \)
\[ d(7!) = d(2^2 \cdot 3 \cdot 5 \cdot 7) = 60 < 120 = 5!. \]

We assume that \( d(n!) < (n - 2)!. \)
\[ d((n+1)!) = d(n!(n + 1)) \leq d(n!) \cdot d(n + 1) < (n - 2)! \cdot d(n + 1) < (n-2)! \cdot (n - 1) = (n - 1)!, \]
because in accordance with Lemma 1, \( d(n + 1) < n - 1. \)
Proposition. The series \( \sum_{n=2}^{\infty} \frac{1}{S(n)!} \) is convergent to a number \( s \in (0.717, 1.253) \), that we call the first constant constant of Smarandache.

Proof. From Lemma 2 it results that \( a(n) < (n - 2)! \), so \( \frac{a(n)}{n!} < \frac{1}{n(n-1)} \) for every \( n \in \mathbb{N} \). 

For \( n \geq 7 \) and \( \sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^{\infty} \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)!} \).

Therefore \( \sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^5 - n} \). \( \tag{7} \)

Because \( \sum_{n=2}^{\infty} \frac{1}{n^5 - n} = 1 \) we have: it exists the number \( s > 0 \), that we call the Smarandache constant, \( s = \sum_{n=2}^{\infty} \frac{1}{S(n)!} \).

From (7) we obtain:

\[
\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^1 - 2} - \frac{1}{3^2 - 3} - \frac{1}{4^2 - 4} + \\
\quad + \frac{1}{5^2 - 5} + \frac{1}{6^2 - 6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1.253.
\]

But, because \( S(n) = n \) for every \( n \in \mathbb{N}^* \), it results:

\[
\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.
\]

Consequently, for this first constant we obtain the framing \( e - 2 < s < 1.253 \), i.e., \( 0.717 < s < 1.253 \).

REFERENCES
