Vertex Graceful Labeling-Some Path Related Graphs

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Abstract: In this article, we show that an algorithm for VG of a caterpillar and proved that \( A(m_j, n) \) is vertex graceful if \( m_j \) is monotonically increasing, \( 2 \leq j \leq n \), when \( n \) is odd, \( 1 \leq m_2 \leq 3 \) and \( m_1 < m_2 \), \( (m_j, n) \cup P_3 \) is vertex graceful if \( m_j \) is monotonically increasing, \( 2 \leq j \leq n \), when \( n \) is odd, \( 1 \leq m_2 \leq 3 \), \( m_1 < m_2 \) and \( C_n \cup C_{n+1} \) is vertex graceful if and only if \( n \geq 4 \).

Key Words: Vertex graceful graphs, vertex graceful labeling, caterpillar, actinia graphs, Smarandachely vertex \( m \)-labeling.

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§1. Introduction

A graph \( G \) with \( p \) vertices and \( q \) edges is said to be vertex graceful if a labeling \( f : V(G) \to \{1,2,3\cdots,p\} \) exists in such a way that the induced labeling \( f^+ : E(G) \to Z_q \) defined by \( f^+((u,v)) = f(u) + f(v)(mod \ q) \) is a bijection. The concept of vertex graceful (VG) was introduced by Lee, Pan and Tsai in 2005. Generally, if replacing \( q \) by an integer \( m \) and \( f^S : E(G) \to Z_m \) also is a bijection, such a labeling is called a Smarandachely vertex \( m \)-labeling. Thus a vertex graceful is in fact a Smarandachely vertex \( q \)-labeling.

All graphs in this paper are finite simple graphs with no loops or multiple edges. The symbols \( V(G) \) and \( E(G) \) denote the vertex set and edge set of the graph \( G \). The cardinality of the vertex set is called the order of \( G \). The cardinality of the edge set is called the size of \( G \). A graph with \( p \) vertices and \( q \) edges is called a \((p, q)\) graph.

§2. Main Results

Algorithm 2.1

1. Let \( v_1, v_2 \cdots v_n \) be the vertices of a path in the caterpillar. (refer Figure 1).
2. Let \( v_j \) be the vertices, which are adjacent to \( v_i \) for \( 1 \leq i \leq n \) and for any \( j \).
3. Draw the caterpillar as a bipartite graph in two partite sets denoted as \( \text{Left (L)} \) which

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contains \( v_1, v_{2j}, v_3, v_{4j}, \ldots \) and for any \( j \) and Right (R) which contains \( v_{1j}, v_2, v_{3j}, v_4, \ldots \) and for any \( j \). (refer Figure 2).

4. Let the number of vertices in \( L \) be \( x \).

5. Number the vertices in \( L \) starting from top down to bottom consecutively as \( 1, 2, \ldots, x \).

6. Number the vertices in \( R \) starting from top down to bottom consecutively as \( (x + 1), \ldots, q \). Note that these numbers are the vertex labels.

7. Compute the edge labels by adding them modulo \( q \).

8. The resulting labeling is vertex graceful labeling.

\[
\begin{align*}
&\text{Figure 1: A caterpillar} \\
&\text{Figure 2: A caterpillar as bipartite graph}
\end{align*}
\]

**Definition 2.2** The graph \( A(m, n) \) obtained by attaching \( m \) pendant edges to the vertices of the cycle \( C_n \) is called Actinia graph.

**Theorem 2.3** A graph \( A(m_j, n), m_j \) is monotonically increasing with difference one, \( 2 \leq j \leq n \) is vertex graceful, \( 1 \leq m_2 \leq 3 \) when \( n \) is odd.

**Proof** Let the graph \( G = A(m_j, n), m_j \) be monotonically increasing with difference one, \( 2 \leq j \leq n \), \( n \) be odd with \( p = n + m_n(\frac{m_n+1}{2}) - m_1(\frac{m_1+1}{2}) = m_2 - 1 \) vertices and \( q = p \) edges. Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of the cycle \( C_n \). Let \( v_{ij}(j = 1, 2, 3, \ldots, n) \) denote the vertices which are adjacent to \( v_i \). By definition of vertex graceful labeling, the required
vertices labeling are

\[
v_i = \begin{cases} 
\left(\frac{i-1}{2}\right)\left(m_2 + \frac{(i+1)}{2}\right) + 1, 1 \leq i \leq n, & i \text{ is odd,} \\
\left(m_2 + 1\right)\left(\frac{n+1}{2}\right) + \frac{(n-2)}{2} + \frac{(i-2)}{2}\left(m_2 + \frac{1}{2}\right) + \frac{1}{2}, 1 \leq i \leq n, & i \text{ is even.}
\end{cases}
\]

\[
v_{ij} = \begin{cases} 
\left(\frac{n-1}{2}\right)\left(m_2 + \frac{(n+1)}{2}\right) + \frac{i-1}{2}\left(m_2 + \frac{3}{2}\right) + \frac{i+1}{2} + j, 1 \leq j \leq m_2 + i - 1, i \text{ is odd;} \\
\left(\frac{i-2}{2}\right)\left(m_2 + \frac{i-2}{2}\right) + \frac{j}{2} + j, 1 \leq i \leq m_2 + i - 1, i \text{ is even.}
\end{cases}
\]

The corresponding edge set labels are as follows:

Let \( A = \{e_i = v_iv_{i+1}/1 \leq i \leq n - 1 \cup e_n = v_nv_1\} \), where

\[
e_i = \left[\left(m_2 + 1\right)\left(n+1\right)\right] + \left(n-1\right) + m_2(i-1) + \frac{i(i+1)}{2} + 1 \pmod{q}
\]

for \( 1 \leq i \leq n \). \( B = \{e_{ij} = v_iv_{ij}/1 \leq i \leq n\} \), where

\[
e_{ij} = \left[\left(n-1\right)\left(m_2 + \frac{(n+1)}{2}\right) + (i-1)\left(m_2 + \frac{i-1}{2}\right) + \frac{i+1}{2} + j + 1 \pmod{q}
\]

for \( 1 \leq i \leq n \) and \( i \) is odd, \( j = 1, 2, \ldots, m_2 + i - 1 \). \( C = \{e_{ij} = v_iv_{ij}/1 \leq i \leq n\} \), where

\[
e_{ij} = \left[\left(m_2 + 1\right)\left(n+1\right)\right] + \left(n-1\right) + \frac{i-2}{2}(2m_2 + i - 1) + i + j \pmod{q}
\]

for \( 1 \leq i \leq n \) and \( i \) is even, \( j = 1, 2, \ldots, m_2 + i - 1 \).

Hence, the induced edge labels of \( G \) are \( q \) distinct integers. Therefore, the graph \( G = A(m_j,n) \) is vertex graceful for \( n \) is odd, and \( m \geq 1 \).

\[ \square \]

**Theorem 2.4** A graph \( A(m_j,n) \cup P_3, m_j \) be monotonically increasing, \( 2 \leq j \leq n \) is vertex graceful, \( 1 \leq m_2 \leq 3, n \) is odd.

**Proof** Let the graph \( G = A(m_j,n) \cup P_3, m_j \) be monotonically increasing, \( 2 \leq j \leq n \), \( n \) is odd with \( p = n + 3 + m_2(m_{n+1}) - m_1(m_{n+1}), m_1 < m_2 \) vertices and \( q = p - 1 \) edges. Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of the cycle \( C_n \). Let \( v_{ij}(j = 1, 2, 3, \ldots, n) \) denote the vertices which are adjacent to \( u_i \). Let \( u_1, u_2, u_3 \) be the vertices of the path \( P_3 \). By definition of vertex graceful labeling, the required vertices labeling are

\[
v_i = \begin{cases} 
\left(\frac{i-1}{2}\right)\left(m_2 + \frac{i+1}{2}\right) + 1, 1 \leq i \leq n, & i \text{ is odd;} \\
\left(m_2 + 1\right)\left(\frac{n+1}{2}\right) + \frac{(n-2)}{2} + \frac{(i-2)}{2}\left(m_2 + \frac{1}{2}\right) + \frac{1}{2}, 1 \leq i \leq n, & i \text{ is even.}
\end{cases}
\]

\[
v_{ij} = \begin{cases} 
\left(\frac{n-1}{2}\right)\left(m_2 + \frac{n+1}{2}\right) + \frac{i-1}{2}\left(m_2 + \frac{3}{2}\right) + \frac{i+1}{2} + j, 1 \leq j \leq m_2 + i - 1, i \text{ is odd;} \\
\left(\frac{i-2}{2}\right)\left(m_2 + \frac{i-2}{2}\right) + \frac{j}{2} + j, 1 \leq i \leq m_2 + i - 1, i \text{ is even.}
\end{cases}
\]

\[
u_i = \frac{n+1}{2}\left(m_2 + \frac{n+1}{2}\right) + \frac{i+1}{2} \text{ for } i = 1, 3 \text{ and } u_2 = p.
\]

The corresponding edge labels are as follows:

Let \( A = \{e_i = v_iv_{i+1}/1 \leq i \leq n - 1 \cup e_n = v_nv_1\} \), where

\[
e_i = \left[\left(m_2 + 1\right)\left(n+1\right)\right] + \left(n-1\right) + m_2(i-1) + \frac{i(i+1)}{2} + 3 \pmod{q}
\]
for $1 \leq i \leq n$. $B = \{e_{ij} = v_iv_{ij}/1 \leq i \leq n\}$, where
\[
e_{ij} = \left[\frac{(n-1)}{2}\left(m_2 + \frac{(n+1)}{2}\right) + (i-1)\left(m_2 + \frac{i-1}{2}\right) + \frac{(i+1)}{2} + j + 3\right] \pmod{q}
\]
for $1 \leq i \leq n$ and $i$ is odd, $j = 1, 2, \cdots, m_2 + i - 1$. $C = \{e_{ij} = v_iv_{ij}/1 \leq i \leq n\}$, where
\[
e_{ij} = \left[\frac{(2m_2 + i - 1)}{2} + \frac{(m_2 + n + 1) + i + j}{2}\right] \pmod{q}
\]
for $1 \leq i \leq n$ and $i$ is even, $j = 1, 2, \cdots, m_2 + i - 1$. $D = \{e_i = u_iu_{i+1} \text{ for } i = 1, 2\}$, where
\[
e_i = \left[\frac{n-1}{2} - 1\right] \pmod{q}
\]
for $i = 1, 2$. Hence, the induced edge labels of $G$ are $q$ distinct integers. Therefore, the graph $G = A(m_j, n) \cup P_3$ is vertex graceful for $n$ is odd.

**Definition 2.5** A regular lobster is defined by each vertex in a path is adjacent to the path $P_2$.

**Theorem 2.6** A regular lobster is vertex graceful.

**Proof** Let $G$ be a 1- regular lobster with $3n$ vertices and $q = 3n - 1$ edges. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of a path $P_n$. Let $v_i$ be the vertices, which are adjacent to $v_i^1$ and $v_i^2$, adjacent to $v_i^1$ for $1 \leq i \leq n$ and $n$ is even. The theorem is proved by two cases. By definition of Vertex graceful labeling, the required vertices labeling are

**Case 1** $n$ is even

$v_i = \begin{cases} 
\frac{3i-1}{2} & 1 \leq i \leq n, i \text{ is odd,} \\
\frac{3(n+i)}{2} & 1 \leq i \leq n, i \text{ is even.}
\end{cases}$

$v_{i1} = \begin{cases} 
\frac{3(n+i)}{2} & 1 \leq i \leq n, i \text{ is odd} \\
\frac{3i-2}{2} & 1 \leq i \leq n, i \text{ is even.}
\end{cases}$

$v_{i2} = \begin{cases} 
\frac{3(i-1)}{2} + 2 & 1 \leq i \leq n, i \text{ is odd,} \\
\frac{3(n+i)}{2} - 1 & 1 \leq i \leq n, i \text{ is even.}
\end{cases}$

The corresponding edge labels are as follows:

Let $A = \{e_i = v_i v_{i+1}/1 \leq i \leq n-1\}$, where $e_i = \left(\frac{3(n+2i)}{2} + 1\right) \pmod{q}$ for $1 \leq i \leq n-1$,

$B = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\frac{3(n+2i)}{2} - 1\right) \pmod{q}$ for $1 \leq i \leq n$ and $i$ is odd,

$C = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\frac{3(n+2i)}{2}\right) \pmod{q}$ for $1 \leq i \leq n$ and is even,

$D = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\frac{3(n+2i)}{2}\right) \pmod{q}$ for $1 \leq i \leq n$ and $i$ is odd,

$E = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\frac{3(n+2i)}{2} - 1\right) \pmod{q}$ for $1 \leq i \leq n$ and is even.
Case 2 \( n \) is odd

\[ v_i = \begin{cases} 
\frac{3i - 1}{2} : & 1 \leq i \leq n, \text{ } i \text{ is odd}, \\
\frac{3(n + i) + 1}{2} : & 1 \leq i \leq n, \text{ } i \text{ is even},
\end{cases} \]

\[ v_{i1} = \begin{cases} 
\frac{3(n + i)}{2} : & 1 \leq i \leq n, \text{ } i \text{ is odd}, \\
\frac{3(i - 2)}{2} + 3 : & 1 \leq i \leq n, \text{ } i \text{ is even},
\end{cases} \]

\[ v_{i2} = \begin{cases} 
\frac{3(i - 1)}{2} + 2 : & 1 \leq i \leq n, \text{ } i \text{ is odd}, \\
\frac{3(n + i - 1)}{2} + 1 : & 1 \leq i \leq n, \text{ } i \text{ is even}.
\end{cases} \]

The corresponding edge labels are determined by \( A = \{ e_i = v_i v_{i+1} / 1 \leq i \leq n - 1 \} \), where \( e_i = \left( \frac{3(n + 2i + 1)}{2} \right) \text{ (mod } q\text{) for } 1 \leq i \leq n - 1 \), \( B = \{ e_{i1} = v_i v_{i1} / 1 \leq i \leq n \} \), where \( e_{i1} = \left( \frac{3(n + 2i) - 1}{2} \right) \text{ (mod } q\text{) for } 1 \leq i \leq n \) and \( i \) is odd, \( C = \{ e_{i1} = v_i v_{i1} / 1 \leq i \leq n \} \), where \( e_{i1} = \left( \frac{3(n + 2i) + 1}{2} \right) \text{ (mod } q\text{) for } 1 \leq i \leq n \) and \( i \) is even, \( D = \{ e_{i2} = v_i v_{i2} / 1 \leq i \leq n \} \), where \( e_{i2} = \left( \frac{3n + 2i + 1}{2} \right) \text{ (mod } q\text{) for } 1 \leq i \leq n \) and \( i \) is odd, \( E = \{ e_{i2} = v_i v_{i2} / 1 \leq i \leq n \} \), where \( e_{i2} = \left( \frac{3(n + 2i) - 1}{2} \right) \text{ (mod } q\text{) for } 1 \leq i \leq n \) and is even. Hence the induced edge labels of \( G \) are \( q \) distinct edges. Therefore, the graph \( G \) is vertex graceful. \( \Box \)

**Theorem 2.7** \( C_n \cup C_{n+1} \) is vertex graceful if and only if \( n \geq 4 \).

**Proof** Let \( G = C_n \cup C_{n+1} \) with \( p = 2n + 1 \) vertices and \( q = 2n + 1 \) edges. Suppose that the vertices of the cycle \( C_n \) run consecutively \( u_1, u_2, \ldots, u_n \) with \( u_n \) joined to \( u_1 \) and that the vertices of the cycle \( C_{n+1} \) run consecutively \( v_1, v_2, \ldots, v_{n+1} \) with \( v_{n+1} \) joined to \( v_1 \).

By definition of vertex graceful labeling

(a) \( u_1 = 1, u_n = 2, u_i = 2i \) for \( i = 2, 3, \ldots, [(n + 1)/2] \), \( u_j = 2(n - j) + 3 \) for \( j = [(n + 3)/2], \ldots, n - 1 \).

(b) \( v_1 = 2, v_2 = 2n - 1 \) and

(i) \( v_{3s+t} = 2n - 4t - 6s + 7, t = 0, 1, 2, s = 1, 2, \ldots, [(n + 1 - 3t)/6] \text{ if } s = \left\lfloor \frac{n + 1 - 3t}{6} \right\rfloor < 1 \text{ then no } s \).

(ii) Write \( \alpha(0) = 0, \alpha(1) = 4, \alpha(2) = 2, \beta(0) = 0, \beta(1) = 3 = \beta(2) \)

\( v_{n+1-3s-t} = 2n - 6s - \alpha(t), t = 0, 1, 2, s = 0, 1, \ldots, \left\lfloor \frac{n - 5 - \beta(t)}{6} \right\rfloor . \text{ If } s = \left\lfloor \frac{n - 5 - \beta(t)}{6} \right\rfloor < 0 \text{ then no } s \text{ value exists} \).

(iii) We consider as that \( v_i \) to \( f(i) \); and suppose that \( n - 2 = 3 \text{ mod}(3), 0 \leq \theta \leq 2 \). There are \( 2 + \theta \) vertices as yet unlabeled. These middle vertices are labeled according to congruence class of modulo 6.
Congruence class | \( n = 0 \text{ (mod 6)} \) | \( f((n + 2)/2) = n + 2, f((n + 4)/2) = n + 3, f((n + 6)/2) = n + 4 \) |
<table>
<thead>
<tr>
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<tbody>
<tr>
<td></td>
<td>( n = 1 \text{ (mod 6)} )</td>
<td>( f((n + 1)/2) = n + 2, f((n + 3)/2) = n + 3, f((n + 5)/2) = n + 4, f((n + 7)/2) = n + 5 )</td>
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<tr>
<td></td>
<td>( n = 2 \text{ (mod 6)} )</td>
<td>( f((n + 2)/2) = n + 2, f((n + 4)/2) = n + 3 )</td>
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<tr>
<td></td>
<td>( n = 2 \text{ (mod 6)} )</td>
<td>( f((n + 1)/2) = n + 4, f((n + 3)/2) = n + 3, f((n + 5)/2) = n + 2 )</td>
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<td></td>
<td>( n = 4 \text{ (mod 6)} )</td>
<td>( f((n + 2)/2) = n + 5, f((n + 3)/2) = n + 4, f((n + 4)/2) = n + 3, f((n + 5)/2) = n + 2 )</td>
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<td>( n = 4 \text{ (mod 6)} )</td>
<td>( f((n + 3)/2) = n + 3, f((n + 5)/2) = n + 2 )</td>
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To check that \( f \) is vertex graceful is very tedious. But we can give basic idea. The \( C_n \) cycle has edges with labels \( \{2k+2/k = 4, 5, \ldots, n-1\} \cup \{0, 3, 5, 7\} \). In this case all the labeling of the edges of the cycle \( C_{n+1} \) run consecutively \( v_1v_2 \) as follows:

1, \((2n-1, 2n-3), (2n-11, 2n-13, 2n-15), \ldots, (2n+1-12k, 2n-1-12k, 2n-3-12k), \ldots\), middle labels, \( \ldots, (2n+3-12k, (2n+5-12k, (2n+7-12k), \ldots, (2n-21, 2n-19, 2n-17), (2n-9, 2n-7, 2n-5), 2 \). The middle labels depend on the congruence class modulo and are best summarized in the following table. If \( n \) is small the terms in brackets alone occur.

<table>
<thead>
<tr>
<th>Congruence class</th>
<th>( n = 0 \text{ (mod 6)} )</th>
<th>( \ldots, (11, 9), 6, 4, 7, (13, 15, 17) \ldots )</th>
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<tbody>
<tr>
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<td>( n = 1 \text{ (mod 6)} )</td>
<td>( \ldots, (13, 11), 6, 4, 7, (13, 15, 17) \ldots )</td>
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<td>( n = 2 \text{ (mod 6)} )</td>
<td>( \ldots, (11), 6, 4, 7, (9) \ldots )</td>
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<tr>
<td></td>
<td>( n = 2 \text{ (mod 6)} )</td>
<td>( \ldots, (13), 7, 4, 6, (9, 11) \ldots )</td>
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<tr>
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<td>( n = 4 \text{ (mod 6)} )</td>
<td>( \ldots, (15, 9), 6, 4, 7(11, 13) \ldots )</td>
</tr>
<tr>
<td></td>
<td>( n = 4 \text{ (mod 6)} )</td>
<td>( \ldots, (9), 7, 6, 4(11, 13, 15) \ldots )</td>
</tr>
</tbody>
</table>

Thus, all these edge labelings are distinct. \( \square \)

References