About Smarandache-Multiplicative Functions

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The main objective of this note is to introduce the notion of the S-multiplicative function and to give some simple properties concerning it. The name of S-multiplicative is short for Smarandache-multiplicative and reflects the main equation of the Smarandache function.

Definition 1. A function $f: \mathbb{N}^* \rightarrow \mathbb{N}^*$ is called S-multiplicative if:

1. $(a,b) = 1 \Rightarrow f(a \cdot b) = \max \{ f(a), f(b) \}$

The following functions are obviously S-multiplicative:

1. The constant function $f: \mathbb{N}^* \rightarrow \mathbb{N}^*, f(n) = 1$.
2. The Erdos function $f: \mathbb{N}^* \rightarrow \mathbb{N}^*, f(n) = \max \{ \pi | p \text{ is prime and } n \vdots p \}$. [1].
3. The Smarandache function $S: \mathbb{N}^* \rightarrow \mathbb{N}, S(n) = \max \{ p | p! : n \}$. [3].

Certainly, many properties of multiplicative functions[2] can be translated for S-multiplicative functions. The main important property of this function is presented in the following.

Definition 2. If $f: \mathbb{N}^* \rightarrow \mathbb{N}$ is a function, then $\overline{f}: \mathbb{N}^* \rightarrow \mathbb{N}$ is defined by

$\overline{f}(n) = \min \{ f(d) | n \vdots d \}$.

Theorem 1. If $f$ is S-multiplicative function, then $\overline{f}$ is S-multiplicative.

Proof. This proof is made using the following simple remark:

(2). $(d_1 \cdot d_2 \wedge (a,b) = 1) \Rightarrow (\exists d_1 | a)(\exists d_2 | b)(d_1, d_2) = 1 \wedge d = (d_1 \cdot d_2)$

If $d_1$ and $d_2$ satisfy (2), then $f(d_1 \cdot d_2) = \max \{ f(d_1), f(d_2) \}$.

Let $a, b$ be two natural numbers, such that $(a,b) = 1$. Therefore, we have

(3) $\overline{f}(a \cdot b) = \min_{d|a \cdot b} \{ f(d_1 \cdot d_2) = \min_{d_1 | a, d_2 | b} \max \{ f(d_1), f(d_2) \} \}.$

Applying the distributing property of the max and min functions, equation (3) is transformed as follows:

$\overline{f}(a \cdot b) = \max_{d_1 | a, d_2 | b} \min \{ f(d_1), f(d_2) \} = \max \{ \overline{f}(a), \overline{f}(b) \}$. Therefore, the function $\overline{f}$ is S-multiplicative.

We believe that many other properties can be deduced for S-multiplicative functions. Therefore, it will be in our attention to further investigate these functions.

References
