Signed Graph Equation $L^K(S) \sim S$

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Abstract: A Smarandachely $k$-signed graph (Smarandachely $k$-marked graph) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$), where $G = (V, E)$ is a graph called the underlying graph of $S$ and $\sigma : E \to (e_1, e_2, \cdots, e_k)$ ($\mu : V \to (\overline{e_1}, \overline{e_2}, \cdots, \overline{e_k})$) is a function, where each $e_i \in \{+,-\}$. Particularly, a Smarandachely 2-signed graph or 2-marked graph is called abbreviated to a signed graph or a marked graph. We characterize signed graphs $S$ for which $L(S) \sim \overline{S}$, $\overline{S} \sim C_E(S)$ and $L^k(S) \sim \overline{S}$, where $\sim$ denotes switching equivalence and $L(S)$, $\overline{S}$ and $C_E(S)$ are denotes line signed graph, complementary signed graph and common-edge signed graph of $S$ respectively.

Key Words: Smarandachely $k$-signed graph, Smarandachely $k$-marked graph, signed graphs, balance, switching, line signed graph, complementary signed graph, common-edge signed graph.

AMS(2000): 05C22.

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [7]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A Smarandachely $k$-signed graph (Smarandachely $k$-marked graph) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$), where $G = (V, E)$ is a graph called the underlying graph of $S$ and $\sigma : E \to (\overline{e_1}, \overline{e_2}, \cdots, \overline{e_k})$ ($\mu : V \to (\overline{e_1}, \overline{e_2}, \cdots, \overline{e_k})$) is a function, where each $e_i \in \{+,-\}$. Particularly, a Smarandachely 2-signed graph or 2-marked graph is called abbreviated to a signed graph or a marked graph. A signed graph $S = (G, \sigma)$ is balanced if every cycle in $S$ has an even number of negative edges (See [8]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of $S$ is positive.

A marking of $S$ is a function $\mu : V(G) \to \{+,-\}$; A signed graph $S$ together with a marking

\[1\text{Received Oct.8, 2009. Accepted Dec. 10, 2009.}\]
μ is denoted by \( S_\mu \).

The following characterization of balanced signed graphs is well known.

**Proposition 1** (E. Sampathkumar [10]) A signed graph \( S = (G, \sigma) \) is balanced if, and only if, there exist a marking \( \mu \) of its vertices such that each edge \( uv \) in \( S \) satisfies \( \sigma(uv) = \mu(u)\mu(v) \).

Behzad and Chartrand [4] introduced the notion of line signed graph \( L(S) \) of a given signed graph \( S \) as follows: \( L(S) \) is a signed graph such that \((L(S))^n \cong L(S^n)\) and an edge \( e_i,e_j \) in \( L(S) \) is negative if, and only if, both \( e_i \) and \( e_j \) are adjacent negative edges in \( S \). Another notion of line signed graph introduced in [6], is as follows: The line signed graph of a signed graph \( S = (G, \sigma) \) is a signed graph \( L(S) = (L(G), \sigma') \), where for any edge \( ee' \) in \( L(S) \), \( \sigma'(ee') = \sigma(e)\sigma(e') \) (see also, E. Sampathkumar et al. [11]. In this paper, we follow the notion of line signed graph defined by M. K. Gill [6].

**Proposition 2** For any signed graph \( S = (G, \sigma) \), its line signed graph \( L(S) = (L(G), \sigma') \) is balanced.

**Proof** We first note that the labeling \( \sigma \) of \( S \) can be treated as a marking of vertices of \( L(S) \). Then by definition of \( L(S) \) we see that \( \sigma'(ee') = \sigma(e)\sigma(e') \) for every edge \( ee' \) of \( L(S) \) and hence, by proposition-1, the result follows.

**Remark:** In [2], M. Acharya has proved the above result. The proof given here is different from that given in [2].

For any positive integer \( k \), the \( k^{\text{th}} \) iterated line signed graph, \( L^k(S) \) of \( S \) is defined as follows:

\[
L^0(S) = S, \quad L^k(S) = L(L^{k-1}(S))
\]

**Corollary** For any signed graph \( S = (G, \sigma) \) and for any positive integer \( k \), \( L^k(S) \) is balanced.

Let \( S = (G, \sigma) \) be a signed graph. Consider the marking \( \mu \) on vertices of \( S \) defined as follows: each vertex \( v \in V \), \( \mu(v) \) is the product of the signs on the edges incident at \( v \). Complement of \( S \) is a signed graph \( \overline{S} = (\overline{G}, \sigma') \), where for any edge \( e = uv \in \overline{G} \), \( \sigma'(uv) = \mu(u)\mu(v) \). Clearly, \( \overline{S} \) as defined here is a balanced signed graph due to Proposition 1.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking \( \mu \) of a signed graph \( S \). Switching \( S \) with respect to a marking \( \mu \) is the operation of changing the sign of every edge of \( S \) to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by \( S_\mu(S) \) and is called \( \mu \)-switched signed graph or just switched signed graph. Two signed graphs \( S_1 = (G, \sigma) \) and \( S_2 = (G', \sigma') \) are said to be isomorphic, written as \( S_1 \cong S_2 \) if there exists a graph isomorphism \( f : G \rightarrow G' \) (that is a bijection \( f : V(G) \rightarrow V(G') \)) such that if \( uv \) is an edge in \( G \) then \( f(u)f(v) \) is an edge in \( G' \) such that for any edge \( e \in G \), \( \sigma(e) = \sigma'(f(e)) \).

Further, a signed graph \( S_1 = (G, \sigma) \) switches to a signed graph \( S_2 = (G', \sigma') \) (or that \( S_1 \) and \( S_2 \) are switching equivalent) written \( S_1 \sim S_2 \), whenever there exists a marking \( \mu \) of \( S_1 \) such that \( S_\mu(S_1) \cong S_2 \). Note that \( S_1 \sim S_2 \) implies that \( G \cong G' \), since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.
Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be \textit{weakly isomorphic} (see [14]) or \textit{cycle isomorphic} (see [15]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle $Z$ in $S_1$ equals to the sign of $\phi(Z)$ in $S_2$. The following result is well known (See [15]).

**Proposition 3** (T. Zaslavasky [15]) Two signed graphs $S_1$ and $S_2$ with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Switching Equivalence of Iterated Line Signed Graphs and Complementary Signed Graphs

In [12], we characterized signed graphs that are switching equivalent to their line signed graphs and iterated line signed graphs. In this paper, we shall solve the equation $L^k(S) \sim \overline{S}$.

We now characterize signed graphs whose complement and line signed graphs are switching equivalent. In the case of graphs the following result is due to Aigner [3] (See also [13] where $H \circ K$ denotes the corona of the graphs $H$ and $K$ [7].

**Proposition 4** (M. Aigner [3]) The line graph $L(G)$ of a graph $G$ is isomorphic with $\overline{G}$ if, and only if, $G$ is either $C_5$ or $K_3 \circ K_1$.

**Proposition 5** For any signed graph $S = (G, \sigma)$, $L(S) \sim \overline{S}$ if, and only if, $G$ is either $C_5$ or $K_3 \circ K_1$.

\textit{Proof} Suppose $L(S) \sim \overline{S}$. This implies, $L(G) \cong \overline{G}$ and hence by Proposition-4 we see that the graph $G$ must be isomorphic to either $C_5$ or $K_3 \circ K_1$.

Conversely, suppose that $G$ is a $C_5$ or $K_3 \circ K_1$. Then $L(G) \cong \overline{G}$ by Proposition-4. Now, if $S$ any signed graph on any of these graphs, By Proposition-2 and definition of complementary signed graph, $L(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. \qed

In [5], the authors define \textit{path graphs} $P_k(G)$ of a given graph $G = (V, E)$ for any positive integer $k$ as follows: $P_k(G)$ has for its vertex set the set $\mathcal{P}_k(G)$ of all distinct paths in $G$ having $k$ vertices, and two vertices in $\mathcal{P}_k(G)$ are adjacent if they represent two paths $P, Q \in \mathcal{P}_k(G)$ whose union forms either a path $P_{k+1}$ or a cycle $C_k$ in $G$.

Much earlier, the same observation as above on the formation of a line graph $L(G)$ of a given graph $G$, Kulli [9] had defined the \textit{common-edge graph} $C_E(G)$ of $G$ as the \textit{intersection graph} of the family $\mathcal{P}_3(G)$ of 2-paths (i.e., paths of length two) each member of which is treated as a set of edges of corresponding 2-path: as shown by him, it is not difficult to see that $C_E(G) \cong L^2(G)$, for any isolate-free graph $G$, where $L(G) := L^1(G)$ and $L^t(G)$ denotes the $t^{th}$ \textit{iterated line graph} of $G$ for any integer $t \geq 2$.

In [12], we extend the notion of $C_E(G)$ to realm of signed graphs: Given a signed graph $S = (G, \sigma)$ its \textit{common-edge signed graph} $C_E(S) = (C_E(G), \sigma')$ is that signed graph whose underlying graph is $C_E(G)$, the common-edge graph of $G$, where for any edge $(e_1e_2, e_2e_3)$ in $C_E(S)$, $\sigma'(e_1e_2, e_2e_3) = \sigma(e_1e_2)\sigma(e_2e_3)$. 
**Proposition 6** (E. Sampathkumar et al. [12]) For any signed graph $S = (G, \sigma)$, its common-edge signed graph $C_E(S)$ is balanced.

We now characterize signed graph whose complement $\overline{S}$ and common-edge signed graph $C_E(S)$ are switching equivalent. In the case of graphs the following result is due to Simic [13].

**Proposition 7** (S. K. Simic [13]) The common-edge graph $C_E(G)$ of a graph $G$ is isomorphic with $\overline{G}$ if, and only if, $G$ is either $C_5$ or $K_2 \circ K_2$.

**Proposition 8** For any signed graph $S = (G, \sigma)$, $\overline{S} \sim C_E(S)$ if, and only if, $G$ is either $C_5$ or $K_2 \circ K_2$.

**Proof** Suppose $\overline{S} \sim C_E(S)$. This implies, $\overline{G} \cong C_E(G)$ and hence by Proposition-7, we see that the graph $G$ must be isomorphic to either $C_5$ or $K_2 \circ K_2$.

Conversely, suppose that $G$ is a $C_5$ or $K_2 \circ K_2$. Then $\overline{G} \cong C_E(G)$ by Proposition-7. Now, if $S$ any signed graph on any of these graphs, By Proposition-6 and definition of complementary signed graph, $C_E(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

We now characterize signed graphs whose complement and its iterated line signed graphs $L^k(S)$, where $k \geq 3$ are switching equivalent. In the case of graphs the following result is due to Simic [13].

**Proposition 9** (S. K. Simic [13]) For any positive integer $k \geq 3$, $L^k(G)$ is isomorphic with $\overline{G}$ if, and only if, $G$ is $C_5$.

**Proposition 10** For any signed graph $S = (G, \sigma)$ and for any positive integer $k \geq 3$, $L^k(S) \sim \overline{S}$ if, and only if, $G$ is $C_5$.

**Proof** Suppose $L^k(S) \sim \overline{S}$. This implies, $L^k(G) \cong \overline{G}$ and hence by Proposition-9 we see that the graph $G$ is isomorphic to $C_5$.

Conversely, suppose that $G$ is isomorphic to $C_5$. Then $L^k(G) \cong \overline{G}$ by Proposition-9. Now, if $S$ any signed graph on $C_5$, By Corollary-2.1 and definition of complementary signed graph, $L^k(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

**References**


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