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Signed Graph Equation $L^K(S) \sim \overline{S}$

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Abstract: A *Smarandachely k-signed graph (Smarandachely k-marked graph)* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-singed graph or 2-marked graph is called abbreviated to a *singed graph* or a *marked graph*. We characterize signed graphs S for which $L(S) \sim \overline{S}$, $\overline{S} \sim C_E(S)$ and $L^k(S) \sim \overline{S}$, where $\sim$ denotes switching equivalence and $L(S)$, $\overline{S}$ and $C_E(S)$ are denotes line signed graph, complementary signed Graph and common-edge signed graph of S respectively.

Key Words: Smarandachely k-signed graph, Smarandachely k-marked graph, signed graphs, balance, switching, line signed graph, complementary signed graph, common-edge signed graph.

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§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [7]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph (Smarandachely k-marked graph)* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-singed graph or 2-marked graph is called abbreviated to a *singed graph* or a *marked graph*. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [8]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking

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μ is denoted by S_μ.

The following characterization of balanced signed graphs is well known.

Proposition 1 (E. Sampathkumar [10]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Behzad and Chartrand [4] introduced the notion of *line signed graph* $L(S)$ of a given signed graph S as follows: $L(S)$ is a signed graph such that $(L(S))^u \cong L(S^u)$ and an edge e_ie_j in $L(S)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [6],is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$ (see also, E. Sampathkumar et al. [11]. In this paper, we follow the notion of line signed graph defined by M. K. Gill [6].

Proposition 2 *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

Proof We first note that the labeling σ of S can be treated as a marking of vertices of $L(S)$. Then by definition of $L(S)$ we see that $\sigma'(ee') = \sigma(e)\sigma(e')$, for every edge ee' of $L(S)$ and hence, by proposition-1, the result follows. □

Remark: In [2], M. Acharya has proved the above result. The proof given here is different from that given in [2].

For any positive integer k, the k^{th} iterated line signed graph, $L^k(S)$ of S is defined as follows:

$$L^0(S) = S,\ L^k(S) = L(L^{k-1}(S))$$

Corollary *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $L^k(S)$ is balanced.*

Let $S = (G, \sigma)$ be a signed graph. Consider the marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident at v. *Complement* of S is a signed graph $\overline{S} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further, a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [14]) or *cycle isomorphic* (see [15]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [15]).

Proposition 3 (T. Zaslavasky [15]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Switching Equivalence of Iterated Line Signed Graphs and Complementary Signed Graphs

In [12], we characterized signed graphs that are switching equivalent to their line signed graphs and iterated line signed graphs. In this paper, we shall solve the equation $L^k(S) \sim \overline{S}$.

We now characterize signed graphs whose complement and line signed graphs are switching equivalent. In the case of graphs the following result is due to Aigner [3] (See also [13] where $H \circ K$ denotes the corona of the graphs H and K [7].

Proposition 4 (M. Aigner [3]) *The line graph $L(G)$ of a graph G is isomorphic with $\overline{G}$ if, and only if, G is either C_5 or $K_3 \circ K_1$.*

Proposition 5 *For any signed graph $S = (G, \sigma)$, $L(S) \sim \overline{S}$ if, and only if, G is either C_5 or $K_3 \circ K_1$.*

Proof Suppose $L(S) \sim \overline{S}$. This implies, $L(G) \cong \overline{G}$ and hence by Proposition-4 we see that the graph G must be isomorphic to either C_5 or $K_3 \circ K_1$.

Conversely, suppose that G is a C_5 or $K_3 \circ K_1$. Then $L(G) \cong \overline{G}$ by Proposition-4. Now, if S any signed graph on any of these graphs, By Proposition-2 and definition of complementary signed graph, $L(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

In [5], the authors define *path graphs* $P_k(G)$ of a given graph $G = (V, E)$ for any positive integer k as follows: $P_k(G)$ has for its vertex set the set $\mathcal{P}_k(G)$ of all distinct paths in G having k vertices, and two vertices in $\mathcal{P}_k(G)$ are adjacent if they represent two paths $P, Q \in \mathcal{P}_k(G)$ whose union forms either a path P_{k+1} or a cycle C_k in G.

Much earlier, the same observation as above on the formation of a line graph $L(G)$ of a given graph G, Kulli [9] had defined the *common-edge graph* $C_E(G)$ of G as the *intersection graph* of the family $\mathcal{P}_3(G)$ of 2-paths (i.e., paths of length two) each member of which is treated as a set of edges of corresponding 2-path; as shown by him, it is not difficult to see that $C_E(G) \cong L^2(G)$, for any isolate-free graph G, where $L(G) := L^1(G)$ and $L^t(G)$ denotes the t^{th} *iterated line graph* of G for any integer $t \geq 2$.

In [12], we extend the notion of $C_E(G)$ to realm of signed graphs: Given a signed graph $S = (G, \sigma)$ its *common-edge signed graph* $C_E(S) = (C_E(G), \sigma')$ is that signed graph whose underlying graph is $C_E(G)$, the common-edge graph of G, where for any edge (e_1e_2, e_2e_3) in $C_E(S)$, $\sigma'(e_1e_2, e_2e_3) = \sigma(e_1e_2)\sigma(e_2e_3)$.

Proposition 6(E. Sampathkumar et al. [12]) *For any signed graph* $S = (G, \sigma)$, *its common-edge signed graph* $C_E(S)$ *is balanced.*

We now characterize signed graph whose complement $\overline{S}$ and common-edge signed graph $C_E(S)$ are switching equivalent. In the case of graphs the following result is due to Simic [13].

Proposition 7(S. K. Simic [13]) *The common-edge graph* $C_E(G)$ *of a graph* G *is isomorphic with* $\overline{G}$ *if, and only if,* G *is either* C_5 *or* $K_2 \circ \overline{K_2}$.

Proposition 8 *For any signed graph* $S = (G, \sigma)$, $\overline{S} \sim C_E(S)$ *if, and only if,* G *is either* C_5 *or* $K_2 \circ \overline{K_2}$.

Proof Suppose $\overline{S} \sim C_E(S)$. This implies, $\overline{G} \cong C_E(G)$ and hence by Proposition-7, we see that the graph G must be isomorphic to either C_5 or $K_2 \circ \overline{K_2}$.

Conversely, suppose that G is a C_5 or $K_2 \circ \overline{K_2}$. Then $\overline{G} \cong C_E(G)$ by Proposition-7. Now, if S any signed graph on any of these graphs, By Proposition-6 and definition of complementary signed graph, $C_E(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

We now characterize signed graphs whose complement and its iterated line signed graphs $L^k(S)$, where $k \geq 3$ are switching equivalent. In the case of graphs the following result is due to Simic [13].

Proposition 9(S. K. Simic [13]) *For any positive integer* $k \geq 3$, $L^k(G)$ *is isomorphic with* $\overline{G}$ *if, and only if,* G *is* C_5.

Proposition 10 *For any signed graph* $S = (G, \sigma)$ *and for any positive integer* $k \geq 3$, $L^k(S) \sim \overline{S}$ *if, and only if,* G *is* C_5.

Proof Suppose $L^k(S) \sim \overline{S}$. This implies, $L^k(G) \cong \overline{G}$ and hence by Proposition-9 we see that the graph G is isomorphic to C_5.

Conversely, suppose that G is isomorphic to C_5. Then $L^k(G) \cong \overline{G}$ by Proposition-9. Now, if S any signed graph on C_5, By Corollary-2.1 and definition of complementary signed graph, $L^k(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

References

[1] R. P. Abelson and M. J. Rosenberg, Symoblic psychologic: A model of attitudinal cognition, *Behav. Sci.*, 3 (1958), 1-13.

[2] M. Acharya, x-Line sigraph of a sigraph, *J. Combin. Math. Combin. Comput.*, 69(2009), 103-111.

[3] M. Aigner, Graph whose complement and line graph are isomorphic, J. Comb. Theory, 7 (1969), 273-275.

[4] M. Behzad and G. T. Chartrand, Line coloring of signed graphs, *Element der Mathematik*, 24(3) (1969), 49-52.

[5] H. J. Broersma and C. Hoede, Path graphs, *J. Graph Theory*, 13(4) (1989), 427-444.

[6] M. K. Gill, *Contributions to some topics in graph theory and its applications*, Ph.D. thesis,

The Indian Institute of Technology, Bombay, 1983.

[7] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[8] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.

[9] V. R. Kulli, On common-edge graphs, *The Karnatak University Journal: Science*-Vol. XVIII(1973), 321-324.

[10] E. Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.

[11] E. Sampathkumar, P. Siva Kota Reddy, and M. S. Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*, to appear.

[12] E. Sampathkumar, P. Siva Kota Reddy, and M. S. Subramanya, Common-edge signed graph of a signed graph, Submitted.

[13] S. K. Simic, Graph equation $L^n(G) \cong \overline{G}$, *univ. Beograd publ. Electrostatic. Fak., Ser. Math Fiz* , 498/541 (1975), 41-44.

[14] T. Sozánsky, Enueration of weak isomorphism classes of signed graphs, *J. Graph Theory*, 4(2)(1980), 127-144.

[15] T. Zaslavsky, Signed Graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.