Singed Total Domatic Number of a Graph

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Abstract: Let \( G \) be a finite and simple graph with vertex set \( V(G) \), \( k \geq 1 \) an integer and let \( f : V(G) \to \{-k, -k+1, \ldots, -1, 1, \ldots, k-1, k\} \) be \( 2k \)-valued function. If \( \sum_{x \in N(v)} f(x) \geq k \) for each \( v \in V(G) \), where \( N(v) \) is the open neighborhood of \( v \), then \( f \) is a Smarandachely \( k \)-Signed total dominating function on \( G \). A set \( \{f_1, f_2, \ldots, f_d\} \) of Smarandachely \( k \)-Signed total dominating function on \( G \) with the property that \( \sum_{i=1}^{d} f_i(x) \leq k \) for each \( x \in V(G) \) is called a Smarandachely \( k \)-Signed total dominating family (function) on \( G \). Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on \( G \). The maximum number of functions in a signed total dominating family on \( G \) is the signed total domatic number of \( G \). In this paper, some properties related signed total domatic number and signed total domination number of a graph are studied and found the sign total domatic number of certain class of graphs such as fans, wheels and generalized Petersen graph.

Key Words: Smarandachely \( k \)-signed total dominating function, signed total domination number, signed total domatic number.

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§1. Terminology and Introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants \([1]\) and \([4]\). We considered finite, undirected, simple graphs \( G = (V, E) \) with vertex set \( V(G) \) and edge set \( E(G) \). The order of \( G \) is given by \( n = |V(G)| \). If \( v \in V(G) \), then the open neighborhood of \( v \) is \( N(v) = \{u \in V(G) | uv \in E(G)\} \) and the closed neighborhood of \( v \) is \( N[v] = \{v\} \cup N(v) \). The number \( d_G(v) = d(v) = |N(v)| \) is the degree of the vertex \( v \in V(G) \), and \( \delta(G) \) is the minimum degree of \( G \). The complete graph and the cycle of order \( n \) are denoted by \( K_n \) and \( C_n \) respectively. A fan and a wheel is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle respectively. The generalized Petersen graph \( P(n, k) \) is defined to be a graph on \( 2n \) vertices with \( V(P(n, k)) = \{v_iu_i : 1 \leq i \leq n\} \) and \( E(P(n, k)) = \).
\{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 1 \leq i \leq n, \text{subscripts modulo } n\}. If \( A \subseteq V(G) \) and \( f \) is a mapping from \( V(G) \) to \( n \) set of numbers, then \( f(A) = \sum_{x \in A} f(x) \).

Let \( k \geq 1 \) be an integer and let \( f : V(G) \to \{-k, k - 1, \cdots, -1, 1, \cdots, k - 1, k\} \) be \( 2k \) valued function. If \( \sum_{x \in N(v)} f(x) \geq k \) for each \( v \in V(G) \), where \( N(v) \) is the open neighborhood of \( v \), then \( f \) is a Smarandachely \( k \)-Signed total dominating function on \( G \). A set \( \{f_1, f_2, \ldots, f_d\} \) of Smarandachely \( k \)-Signed total dominating function on \( G \) with the property that \( \sum_{i=1}^{d} f_i(x) \leq k \) for each \( x \in V(G) \) is called a Smarandachely \( k \)-Signed total dominating family (function) on \( G \). Particularly, a Smarandachely \( 1 \)-Signed total dominating function or family is called signed total dominating function or family on \( G \). The signed total dominating function is defined in [6] as a two valued function \( f : V(G) \to \{-1, 1\} \) such that \( \sum_{x \in N(v)} f(x) \geq 1 \) for each \( v \in V(G) \). The minimum number of weights \( w(f) \), taken over all signed total dominating functions \( f \) on \( G \), is called the signed total domination number \( \gamma_s^+(G) \). Signed total domination has been studied in [3].

A set \( \{f_1, f_2, \ldots, f_d\} \) of signed total dominating functions on \( G \) with the property that \( \sum_{i=1}^{d} f_i(x) \leq 1 \) for each \( x \in V(G) \), is called a signed total dominating family on \( G \). The maximum number of functions in a signed total dominating family is the signed total domatic number of \( G \), denoted by \( d_s^+(G) \). Signed total domatic number was introduced by Guan Mei and Shan Er-fang [2]. Guan Mei and Shan Er-fang [2] have determined the basic properties of \( d_s^+(G) \). Some of them are analogous to those of the signed domatic number in [5] and studied sharp bounds of the signed total domatic number of regular graphs, complete bipartite graphs and complete graphs. Guan Mei and Shan Er-fang [2] presented the following results which are useful in our investigations.

**Proposition 1.1** ([6]) For Circuit \( C_n \) of length \( n \) we have \( \gamma_s^+(C_n) = n \).

**Proof** Here no other signed total dominating exists than the constants equal to 1. \( \square \)

**Theorem 1.2** ([3]) Let \( T \) be a tree of order \( n \geq 2 \). then, \( \gamma_s^+(T) = n \) if and only if every vertex of \( T \) is a support vertex or is adjacent to a vertex of degree 2.

**Proposition 1.3** ([2]) The signed total domatic number \( d_s^+(G) \) is well defined for each graph \( G \).

**Proposition 1.4** ([2]) For any graph \( G \) of order \( n \), \( \gamma_s^+(G) \cdot d_s^+(G) \leq n \).

**Proposition 1.5** ([2]) If \( G \) is a graph with the minimum degree \( \delta(G) \), then \( 1 \leq d_s^+(G) \leq \delta(G) \).

**Proposition 1.6** ([2]) The signed total domatic number is an odd integer.

**Corollary 1.7** ([2]) If \( G \) is a graph with the minimum degree \( \delta(G) = 1 \) or 2, then \( d_s^+(G) = 1 \). In particular, \( d_s^+(C_n) = d_s^+(P_n) = d_s^+(K_{1,n-1}) = d_s^+(T) = 1 \), where \( T \) is a tree.

\section{Properties of the Signed Total Domatic Number}

**Proposition 2.1** If \( G \) is a graph of order \( n \) and \( \gamma_s^+(G) \geq 0 \) then, \( \gamma_s^+(G) + d_s^+(G) \leq n + 1 \) equality
holds if and only if $G$ is isomorphic to $C_n$ or tree $T$ of order $n \geq 2$.

Proof. Let $G$ be a graph of order $n$. The inequality follows from the fact that for any two non-negative integers $a$ and $b$, $a + b \leq ab + 1$. By Proposition 1.4 we have,

$$\gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$$

Suppose that $\gamma_t^s(G) + d_t^s(G) = n + 1$ then, $n + 1 = \gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$.

This implies that $\gamma_t^s(G) + d_t^s(G) = \gamma_t^s(G) \cdot d_t^s(G) + 1$. This shows that $\gamma_t^s(G) \cdot d_t^s(G) = n$ solving equations 1 and 2 simultaneously, we have either $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ or $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$. If $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ then $n = d_t^s(G) \leq \delta(G)$ there fore, $\delta(G) \geq n$ a contradiction.

If $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$ then by Proposition 1.1 and Proposition 1.2, we have $\gamma_t^s(C_n) = n$ and $d_t^s(C_n) = 1$ and By Theorem 1.2, If $T$ is a tree of order $n \geq 2$ then, $\gamma_t^s(T) = n$ if and only if every vertex of $T$ is a support vertex or is adjacent to a vertex of degree 2 and $d_t^s(T) = 1$. □

Theorem 2.2 Let $G$ be a graph of order $n$ then $d_t^s(G) + d_t^s(\hat{G}) \leq n - 1$.

Proof. Let $G$ be a regular graph order $n$, By Proposition 1.5 we have $d_t^s(G) \leq \delta(G)$ and $d_t^s(G) \leq \hat{d_t^s(G)}$. Thus we have,

$$d_t^s(G) + d_t^s(\hat{G}) \leq \delta(G) + \hat{d_t^s(G)} = \delta(G) + (n - 1 - \Delta(G)) \leq n - 1.$$ 

Thus the inequality holds. □

§3. Signed Total Domatic Number of Fans, Wheels and Generalized Petersen Graph

Proposition 3.1 Let $G$ be a fan of order $n$ then $d_t^s(G) = 1$.

Proof. Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be the vertex set of the fan $G$ such that $x_1, x_2, \ldots, x_n, x_1$ is a cycle of length $n$ and $x_n$ is adjacent to $x_i$ for each $i = 2, 3, \ldots, n - 2$. By Proposition 1.5 and Proposition 1.6, $1 \leq d_t^s(G) \leq \delta(G) = 2$, which implies $d_t^s(G) = 1$ which proves the result. □

Proposition 3.2 If $G$ is a wheel of order $n$ then $d_t^s(G) = 1$.

Proof. Let $x_1, x_2, \ldots, x_n$ be the vertex set of the wheel $G$ such that $x_1, x_2, \ldots, x_n, x_1$ is a cycle of length $n - 1$ and $x_n$ is adjacent to $x_i$ for each $i = 1, 2, 3, \ldots, n - 1$. According to the Proposition 1.5 and Proposition 1.6, we observe that either $d_t^s(G) = 1$ or $d_t^s(G) = 3$. Suppose to the contrary that $d_t^s(G) = 3$. Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating family. Because of $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$, there exists at least one function say $f_1$ with $f_1(x_n) = -1$ The condition $\sum_{x \in N(v)} f_1(x) \geq 1$ for each $v \in (V(G) - \{x_n\})$ yields $f_1(x) = 1$ for each some $i \in \{1, 2, \ldots, n - 1\}$ and $t = 2, 3$ then it follows that $f_t(x_i + 1) = f_t(x_i + 2) = 1$, where the indices are taken taken modulo $n - 1$ and $f_t(x_n) = 1$. Consequently, the function $f_t$ has at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ for $n$ is odd and $\frac{n}{2} - 1$ for $n$ is even number of vertices $x \in V(G)$ such that
$f_i(x) = -1$. Thus there exist at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ for $n$ is odd and $\frac{n}{2} - 1$ for $n$ is even number of vertices $x \in V(G)$ such that $f_i(x) = -1$ for at least one $i = 1, 2, 3$. Since $n \geq 4$, we observe that $2\left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) = 2 \left( \frac{n}{2} - 1 \right) + 1 < n$ for $n$ is odd and $2 \left( \frac{n}{2} - 1 \right) + 1 < n$, a contradiction to $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$ for each $x \in V(G)$. \qed

**Proposition 3.3** Let $G = P(n, k)$ be a generalized Petersen graph then for $k = 1, 2$, $d^*_t(G) = 1$.

**Proof** The generalized Petersen graph $P(n, 1)$ is a graph on $2n$ vertices with

$$V(P(n, k)) = \{v_i u_i : 1 \leq i \leq n\}$$

and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+1} : 1 \leq i \leq n, \text{subscripts modulo } n\}$. According to the Proposition 1.5, Proposition 1.6, we observe that $d_1^t(G) = 1$ or $d_1^t(G) = 3$.

**Case 1:** $k = 1$

Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating functions. Because of $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $i \in \{1, 2, \ldots, 2n\}$, there exist at least one number $j \in \{1, 2, 3\}$ such that $f_j(v_i) = -1$. Let, for example, $f_1(v_k) = -1$ for for any $t \in \{1, 2, \ldots, 2n\}$ then $\sum_{x \in N(v_i)} f_1(v) \geq 1$ implies that $f_1(v_k) = f_1(v_{k+1}) = -1$ for $k \equiv 0, 1 \mod 4$ and $f_1(v_k) = -1$ for $k \equiv 0 \mod 3$. This implies, there exist at most $8r, 8r + 2, 8r + 4, 8r + 6, r \geq 1$ vertices such that $f_1(v) = -1$ for each $t = 2, 3$ when $P(n, 1)$ is of order $2(6r + l)$ for $0 \leq l \leq 2, 2(6r + 3), 2(6r + 4), 2(6r + 5)$ respectively. Thus there exist $3(8r) = 3(8(\frac{n}{2} - \frac{1}{6})) < n$ (similarly $n$ for all values of vertex set) a contradiction to $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $v \in V(G)$.

**Case 2:** $k = 2$

Similar to the proof of Case 1, we can prove the claim in this case. \qed

**References**


