Scientia Magna
Vol. 6 (2010), No. 3, 84-88

Smarandachely antipodal signed digraphs

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Abstract A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. In this paper, we define the Smarandachely antipodal signed digraph $\overrightarrow{A}(D)$ of a given signed digraph $S = (D, \sigma)$ and offer a structural characterization of antipodal signed digraphs. Further, we characterize signed digraphs S for which $S \sim \overrightarrow{A}(S)$ and $\overline{S} \sim \overrightarrow{A}(S)$ where $\sim$ denotes switching equivalence and $\overrightarrow{A}(S)$ and $\overline{S}$ are denotes the Smarandachely antipodal signed digraph and complementary signed digraph of S respectively.

Keywords Smarandachely k-signed digraphs, Smarandachely k-marked digraphs, balance, switching, Smarandachely antipodal signed digraphs, negation.

§1. Introduction

For standard terminology and notion in digraph theory, we refer the reader to the classic text-books of Bondy and Murty [1] and Harary et al.[3]; the non-standard will be given in this paper as and when required.

A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. A *signed digraph* is an ordered pair $S = (D, \sigma)$, where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. A *marking* of S is a function $\mu : V(D) \to \{+, -\}$. A signed digraph S together with a marking μ is denoted by S_μ. A signed digraph $S = (D, \sigma)$ is *balanced* if every semicycle of S is positive (Harary et al.[3]). Equivalently, a signed digraph is balanced if every semicycle has an even number of negative arcs. The following characterization of balanced signed digraphs is obtained by E. Sampathkumar et al.[5].

Proposition 1.1.[5] A signed digraph $S = (D, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$.

Let $S = (D, \sigma)$ be a signed digraph. Consider the marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the signs on the arcs incident at v. *Complement* of S is a signed digraph $\overline{S} = (\overline{D}, \sigma')$, where for any arc $e = \overrightarrow{uv} \in \overline{D}$, $\sigma'(\overrightarrow{uv}) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed digraph due to Proposition 1.1.

In [5], the authors define switching and cycle isomorphism of a signed digraph as follows:

Let $S = (D, \sigma)$ and $S' = (D', \sigma')$, be two signed digraphs. Then S and S' are said to be *isomorphic*, if there exists an isomorphism $\phi : D \to D'$ (that is a bijection $\phi : V(D) \to V(D')$ such that if $\overrightarrow{uv}$ is an arc in D then $\overrightarrow{\phi(u)\phi(v)}$ is an arc in D') such that for any arc $\overrightarrow{e} \in D$, $\sigma(\overrightarrow{e}) = \sigma'(\phi(\overrightarrow{e}))$. For switching in signed graphs and some results involving switching refer the paper [4].

Given a marking μ of a signed digraph $S = (D, \sigma)$, *switching* S with respect to μ is the operation changing the sign of every arc $\overrightarrow{uv}$ of S' by $\mu(u)\sigma(\overrightarrow{uv})\mu(v)$. The signed digraph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called μ *switched signed digraph* or just *switched signed digraph*.

Further, a signed digraph S switches to signed digraph S' (or that they are switching equivalent to each other), written as $S \sim S'$, whenever there exists a marking of S such that $\mathcal{S}_\mu(S) \cong S'$.

Two signed digraphs $S = (D, \sigma)$ and $S' = (D', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : D \to D'$ such that the sign $\sigma(Z)$ of every semicycle Z in S equals to the sign $\sigma(\phi(Z))$ in S'.

Proposition 1.2.[4] Two signed digraphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Smarandachely antipodal signed digraphs

In [2], the authors introduced the notion antipodal digraph of a digraph as follows: For a digraph $D = (V, \mathcal{A})$, the *antipodal digraph* $\overrightarrow{A}(D)$ of $D = (V, \mathcal{A})$ is the digraph with $V(\overrightarrow{A}(D)) = V(D)$ and $\mathcal{A}(\overrightarrow{A}(D)) = \{(u, v) : u, v \in V(D) \text{ and } d_D(u, v) = diam(D)\}$.

We extend the notion of $\overrightarrow{A}(D)$ to the realm of signed digraphs. In a signed digraph $S = (D, \sigma)$, where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. The *Smarandachely antipodal signed digraph* $\overrightarrow{A}(S) = (\overrightarrow{A}(D), \sigma')$ of a signed digraph $S = (D, \sigma)$ is a signed digraph whose underlying digraph is $\overrightarrow{A}(D)$ called *antipodal digraph* and sign of any arc $e = \overrightarrow{uv}$ in $\overrightarrow{A}(S)$, $\sigma'(e) = \mu(u)\mu(v)$, where for any $v \in V$, $\mu(v) = \prod_{u \in N(v)} \sigma(uv)$. Further, a signed digraph $S = (D, \sigma)$ is called *Smarandachely antipodal signed digraph*, if $S \cong \overrightarrow{A}(S')$, for some signed digraph S'. The following result indicates the limitations of the notion $\overrightarrow{A}(S)$ as introduced above, since the entire class of unbalanced signed digraphs is forbidden to be antipodal signed digraphs.

Proposition 2.1. For any signed digraph $S = (D, \sigma)$, its Smarandachely antipodal signed graph $A(S)$ is balanced.

Proof. Since sign of any arc $e = \overrightarrow{uv}$ in $\overrightarrow{A}(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1.1, $\overrightarrow{A}(S)$ is balanced.

For any positive integer k, the k^{th} iterated antipodal signed digraph $\overrightarrow{A}(S)$ of S is defined as follows:

$$\overrightarrow{A}^0(S) = S,\ A^k(S) = \overrightarrow{A}(\overrightarrow{A}^{k-1}(S)).$$

Corollary 2.2. For any signed digraph $S = (D, \sigma)$ and any positive integer k, $\overrightarrow{A}^k(S)$ is balanced.

In [2], the authors characterized those digraphs that are isomorphic to their antipodal digraphs.

Proposition 2.3.[2] For a digraph $D = (V, \mathcal{A})$, $D \cong \overrightarrow{A}(D)$ if, and only if, $D \cong K_p^*$.

Proof. First, suppose that $D \cong \overrightarrow{A}(D)$. If $(u, v) \in \mathcal{A}$ then $(u, v) \in \mathcal{A}(\overrightarrow{A}(D))$. Therefore, $d_D(u, v) = 1 = diam(D)$. Since K_p^* is the only digraph of diameter 1, we have $D \cong K_p^*$.

For the converse, if $D \cong K_p^*$, then $diam(D) = 1$ and for every pair u, v of vertices in D, the distance $d_D(u, v) = 1$. Hence, $\overrightarrow{A}(D) \cong K_p^*$ and $D \cong \overrightarrow{A}(D)$.

We now characterize the signed digraphs that are switching equivalent to their Smarandachely antipodal signed graphs.

Proposition 2.4. For any signed digraph $S = (D, \sigma)$, $S \sim \overrightarrow{A}(S)$ if, and only if, $D \cong K_p^*$ and S is balanced signed digraph.

Proof. Suppose $S \sim \overrightarrow{A}(S)$. This implies, $D \cong \overrightarrow{A}(D)$ and hence D is K_p^*. Now, if S is any signed digraph with underlying digraph as K_p^*, Proposition 2.1 implies that $\overrightarrow{A}(S)$ is balanced and hence if S is unbalanced and its $\overrightarrow{A}(S)$ being balanced can not be switching equivalent to S in accordance with Proposition 1.2. Therefore, S must be balanced.

Conversely, suppose that S is an balanced signed digraph and D is K_p^*. Then, since $\overrightarrow{A}(S)$ is balanced as per Proposition 2.1 and since $D \cong \overrightarrow{A}(D)$, the result follows from Proposition 1.2 again.

Proposition 2.5. For any two signed digraphs S and S' with the same underlying digraph, their Smarandachely antipodal signed digraphs are switching equivalent.

Proposition 2.6.[2] For a digraph $D = (V, \mathcal{A})$, $\overline{D} \cong \overrightarrow{A}(D)$ if, and only if,

i) $diam(D) = 2$.

ii) D is not strongly connected and for every pair u, v of vertices of D, the distance $d_D(u, v) = 1$ or $d_D(u, v) = \infty$.

In view of the above, we have the following result for signed digraphs:

Proposition 2.7. For any signed digraph $S = (D, \sigma)$, $\overline{S} \sim \overrightarrow{A}(S)$ if, and only if, D satisfies conditions of Proposition 2.6.

Proof. Suppose that $\overrightarrow{A}(S) \sim \overline{S}$. Then clearly we have $\overrightarrow{A}(D) \cong \overline{D}$ and hence D satisfies conditions of Proposition 2.6.

Conversely, suppose that D satisfies conditions of Proposition 2.6. Then $\overline{D} \cong \overrightarrow{A}(D)$ by Proposition 2.6. Now, if S is a signed digraph with underlying digraph satisfies conditions of Proposition 2.6, by definition of complementary signed digraph and Proposition 2.1, $\overline{S}$ and $\overrightarrow{A}(S)$ are balanced and hence, the result follows from Proposition 1.2.

The notion of *negation* $\eta(S)$ of a given signed digraph S defined in [6] as follows: $\eta(S)$ has the same underlying digraph as that of S with the sign of each arc opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Proposition 2.4 & 2.7 provides easy solutions to two other signed digraph switching equivalence relations, which are given in the following results.

Corollary 2.8. For any signed digraph $S = (D, \sigma)$, $S \sim \overrightarrow{A}(\eta(S))$.

Corollary 2.9. For any signed digraph $S = (D, \sigma)$, $\overline{S} \sim \overrightarrow{A}(\eta(S))$.

Problem. Characterize signed digraphs for which

i) $\eta(S) \sim \overrightarrow{A}(S)$.

ii) $\eta(\overline{S}) \sim \overrightarrow{A}(S)$.

For a signed digraph $S = (D, \sigma)$, the $\overrightarrow{A}(S)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\overrightarrow{A}(S)$ is balanced.

Proposition 2.10. Let $S = (D, \sigma)$ be a signed digraph. If $\overrightarrow{A}(G)$ is bipartite then $\eta(\overrightarrow{A}(S))$ is balanced.

Proof. Since, by Proposition 2.1, $\overrightarrow{A}(S)$ is balanced, if each semicycle C in $\overrightarrow{A}(S)$ contains even number of negative arcs. Also, since $\overrightarrow{A}(D)$ is bipartite, all semicycles have even length; thus, the number of positive arcs on any semicycle C in $\overrightarrow{A}(S)$ is also even. Hence $\eta(\overrightarrow{A}(S))$ is balanced.

§3. Characterization of Smarandachely antipodal signed graphs

The following result characterize signed digraphs which are Smarandachely antipodal signed digraphs.

Proposition 3.1. A signed digraph $S = (D, \sigma)$ is a Smarandachely antipodal signed digraph if, and only if, S is balanced signed digraph and its underlying digraph D is an antipodal graph.

Proof. Suppose that S is balanced and D is a $\overrightarrow{A}(D)$. Then there exists a digraph H such that $\overrightarrow{A}(H) \cong D$. Since S is balanced, by Proposition 1.1, there exists a marking μ of D such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$. Now consider the signed digraph $S' = (H, \sigma')$, where for any arc e in H, $\sigma'(e)$ is the marking of the corresponding vertex in D. Then clearly, $\overrightarrow{A}(S') \cong S$. Hence S is an Smarandachely antipodal signed digraph.

Conversely, suppose that $S = (D, \sigma)$ is a Smarandachely antipodal signed digraph. Then there exists a signed digraph $S' = (H, \sigma')$ such that $\overrightarrow{A}(S') \cong S$. Hence D is the $A(D)$ of H and by Proposition 2.1, S is balanced.

Acknowledgement

The first two authors would like to acknowledge and extend their heartfelt gratitude to Sri. B. Premnath Reddy, Chairman, Acharya Institutes, for his vital encouragement and support.

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