Smarandachely $k$-Constrained Number of Paths and Cycles

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Abstract: A Smarandachely $k$-constrained labeling of a graph $G(V,E)$ is a bijective mapping $f : V \cup E \rightarrow \{1, 2, ... , |V| + |E|\}$ with the additional conditions that $|f(u) - f(v)| \geq k$ whenever $uv \in E$, $|f(u) - f(uv)| \geq k$ and $|f(uv) - f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph $G$ which admits a such labeling is called a Smarandachely $k$-constrained total graph, abbreviated as $k$–CTG. The minimum number of isolated vertices required for a given graph $G$ to make the resultant graph a $k$–CTG is called the $k$-constrained number of the graph $G$ and is denoted by $t_k(G)$. In this paper we settle the open problems 3.4 and 3.6 in [4] by showing that $t_k(P_n) = 0$, if $k \leq k_0; 2(k - k_0)$, if $k > k_0$ and $2n \equiv 1$ or 2 (mod 3); $2(k - k_0) - 1$ if $k > k_0; 2n \equiv 0$(mod 3) and $t_k(C_n) = 0$, if $k \leq k_0; 2(k - k_0)$, if $k > k_0$ and $2n \equiv 0$ (mod 3); $3(k - k_0)$ if $k > k_0$ and $2n \equiv 1$ or 2 (mod 3), where $k_0 = \lfloor \frac{2n - 1}{3} \rfloor$.

Key Words: Smarandachely $k$-constrained labeling, Smarandachely $k$-constrained total graph, $k$-constrained number, minimal $k$-constrained total labeling.

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§1. Introduction

All the graphs considered in this paper are simple, finite and undirected. For standard terminology and notations we refer [1], [3]. There are several types of graph labelings studied by various authors. We refer [2] for the entire survey on graph labeling. In [4], one such labeling called Smarandachely labeling is introduced. Let $G = (V,E)$ be a graph. A bijective mapping $f : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\}$ is called a Smarandachely $k$–constrained labeling of $G$ if it satisfies the following conditions for every $u, v, w \in V$ and $k \geq 2$;

1. $|f(u) - f(v)| \geq k$
2. $|f(u) - f(uv)| \geq k$,  

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Let \( S \) be the set of first \( k \) consecutive positive integers not in \( x \).  

1. Let \( x \) be an edge \( 1 \leq x \). 
2. Lemma \( S \) is an edge whenever \( x \) is incident with \( 2 \). 
3. Let \( f \) be a minimal \( k \)-constrained total labeling of \( G \). Then for each \( l, 1 \leq l \leq k \), determine the value of \( f(l) - f(l_j) \) for some \( l_j \), where \( |f(l_i) - f(l_j)| < k < k \), a contradiction.

Further, if \( f(l_j) \neq i \) for any \( l, j \) with \( 1 \leq l \leq k_0, 1 \leq j \leq 3 \) for some \( i \in S \), then \( i \) should be assigned to an isolated vertex. So, span of \( f \) will increase, hence \( f \) cannot be minimal. \( \square \)

**Problem 1.1** For any integers \( n, k \geq 3 \), determine the value of \( t_k(P_n) \).

**Problem 1.2** For any integers \( n, k \geq 3 \), determine the value of \( t_k(C_n) \).

\section*{2. \( k \)-Constrained Number of a Path}

Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n - 1\} \). Designate the vertex \( v_i \) of \( P_n \) as \( 2i - 1 \) and the edge \( v_jv_{j+1} \) as \( 2j \), for each \( i, 1 \leq i \leq n \) and \( 1 \leq j \leq n - 1 \).

**Lemma 2.1** Let \( k_0 = \left[ \frac{2n - 1}{3} \right] \) and \( S_l = \{3l - 2, 3l - 1, 3l\} \) for \( 1 \leq l \leq k_0 \). Let \( f \) be a minimal \( k \)-constrained total labeling of \( P_n \). Then for each \( i, 1 \leq i \leq k_0 \), there exist a \( l, 1 \leq l \leq k_0 \) and a \( x \in S_l \) such that \( f(x) = i \).

**Proof** For \( 1 \leq l \leq k_0 \), let \( S_l = \{l_1, l_2, l_3\} \), where \( l_1 = 3l - 2, l_2 = 3l - 1, l_3 = 3l \). Let \( S = \{1, 2, 3, \ldots, k_0\} \) and \( S \) be a minimal \( k \)-constrained total labeling of \( P_n \). Then for each \( i, 1 \leq i \leq k_0 + 1 \), otherwise if \( f(l_i), f(l_j) \in S \) for \( 1 \leq i, j \leq 3, i \neq j \), then \( |f(l_i) - f(l_j)| < k_0 < k \), a contradiction.

Further, if \( f(l_j) \neq i \) for any \( l, j \) with \( 1 \leq l \leq k_0, 1 \leq j \leq 3 \) for some \( i \in S \), then \( i \) should be assigned to an isolated vertex. So, span of \( f \) will increase, hence \( f \) cannot be minimal. \( \square \)

**Lemma 2.2** Let \( S_l = \{3l - 2, 3l - 1, 3l\} \) and \( f \) be a minimal \( k \)-constrained total labeling of \( P_n \). Let \( f(x) = s_1 \) and \( f(y) = s_2 \) for some \( x \in S_l \) and \( y \in S_{l+1} \) for some \( l, 1 \leq l < m \leq k_0 \) and \( 1 \leq s_1, s_2 \leq k_0 \), where \( k_0 = \left[ \frac{2n - 1}{3} \right] \). Then \( y = x + 3 \).

**Proof** Let \( x_1, x_2, x_3 \) be the elements of \( S_l \) and \( x_4, x_5, x_6 \) be that of \( S_{l+1} \) (i.e. if \( x_1 \) is a vertex of \( P_n \) then \( x_3, x_5 \) are vertices and \( x_2 \) is an edge \( x_1x_3 \); \( x_4 \) is an edge \( x_3x_5 \) and \( x_6 \) is incident with \( x_5 \) or if \( x_1 \) is an edge, then \( x_1 \) is incident with \( x_2 \); \( x_2, x_4, x_6 \) are vertices and \( x_3 \) is an edge \( x_2x_4 \), \( x_5 \) is an edge \( x_4x_6 \)).

Let \( f \) be a minimal \( k \)-constrained total labeling of \( P_n \) and \( S_1, S_2, \ldots, S_{k_0} \) be the sets as defined in the Lemma 2.1. Let \( S_\alpha \) be the set of first \( k_0 \) consecutive positive integers required for labeling of exactly one element of \( S_l \) for each \( l, 1 \leq l \leq k_0 \) as in Lemma 2.1. Then each set \( S_l, 1 \leq l \leq k_0 \) contains exactly two unassigned elements. Again by Lemma 2.1 exactly one of these unassigned element can be assigned by the set \( S_\beta \) containing next possible \( k_0 \) consecutive positive integers not in \( S_\alpha \). After labeling the elements of the set \( S_l, 1 \leq l \leq k_0 \) by the labels in
$S_a \cup S_b$, each $S_t$ contains exactly one element unassigned. Thus these elements can be assigned as per Lemma 2.1 again by the set $S_\gamma$ having next possible $k_0$ consecutive positive integers not in $S_a \cup S_b$.

Let us now consider two consecutive sets $S_i, S_{i+1}$ (Two sets $S_i$ and $S_j$ are said to be consecutive if they are disjoint and there exists $x \in S_i$ and $y \in S_j$ such that $xy$ is an edge). Let $\alpha_1, \alpha_2 \in S_\alpha, x_i \in S_i$ and $x_j \in S_{i+1}$ such that $f(x_i) = \alpha_1$ and $f(x_j) = \alpha_2$ (such $\alpha_1, \alpha_2, x_i$ and $x_j$ exist by Lemma 2.1). Then, as $f$ is a minimal $k$-constrained total labeling of $P_n$, it follows that $|j - i| > 2$ implies $j \geq i + 3$. Now we claim that $j = i + 3$. We note that if $i = 3$, then the claim is obvious. If $i \neq 3$, then we have the following cases.

Case 1 $i = 1$

If $j \neq 4$ then

Subcase 1 $j = 5$

By Lemma 2.1, there exists $\beta_1, \beta_2 \in S_\beta$ and $x_r \in S_i, x_s \in S_{i+1}$ such that $f(x_r) = \beta_1$ and $f(x_s) = \beta_2$. Now $f(x_1) = \alpha_1$, $f(x_5) = \alpha_2$ implies $r = 2$ or $r = 3$ (i.e. $f(x_2) = \beta_1$ or $f(x_3) = \beta_1$).

Subsubcase 1 $r = 2$ (i.e. $f(x_2) = \beta_1$)

In this case, $f(x_6) = \beta_2$ (since $f(x_2) = \beta_1$ and $f(x_3) = \beta_2$ implies $|j - i| > 2$) and hence by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$ which is inadmissible as $x_3$ and $x_4$ are incident to each other and $|\gamma_1 - \gamma_2| < k_0 < k$.

Subsubcase 2 $r = 3$ (i.e. $f(x_3) = \beta_1$)

Again in this case, $f(x_6) = \beta_2$. So $f(x_2) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$ which is contradiction as $x_2$ and $x_4$ are adjacent to each other and $|\gamma_1 - \gamma_2| < k_0 < k$.

Subcase 2 $j = 6$

Now $f(x_1) = \alpha_1, f(x_6) = \alpha_2$ implies $f(x_2) = \beta_1$ or $f(x_3) = \beta_1$.

Subsubcase 1 $f(x_2) = \beta_1$

In this case, $f(x_5) = \beta_2$ and hence by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$, which is a contradiction as $x_3$ and $x_4$ are incident to each other.

Subsubcase 2 $f(x_3) = \beta_1$

In this case, $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$ none of them is possible.

Thus we conclude in Case 1 that if $i = 1$, then $j = 4$, so $j = i + 3$.

Case 2 $i = 2$

In this case we have $j \geq i + 3$, so $j \geq 5$. If $j \neq 5$ then $j = 6$. Now $f(x_2) = \alpha_1, f(x_6) = \alpha_2$ implies $f(x_1) = \beta_1$ or $f(x_3) = \beta_1$.

Subcase 1: $f(x_1) = \beta_1$

But then $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$.

Subsubcase 1 $f(x_4) = \beta_2$
In this case, \(f(x_4) = \beta_2\) and by Lemma 2.1 \(f(x_3) = \gamma_1, f(x_5) = \gamma_2\), which is a contradiction as \(x_3\) and \(x_5\) are adjacent to each other.

**Subsubcase 2 \(f(x_3) = \beta_2\)**

In this case, \(f(x_5) = \beta_2\) and by Lemma 2.1 \(f(x_3) = \gamma_1, f(x_4) = \gamma_2\), which is not possible as \(x_3\) and \(x_4\) are incident to each other.

**Subcase 2 \(f(x_3) = \beta_1\)**

In this case, \(f(x_4) = \beta_2\) or \(f(x_5) = \beta_2\) none of them is possible.

Thus in this case 2, we conclude that if \(i = 2\), then \(j = 5\), so \(j = i + 3\).

Thus, we conclude that the labels in \(S_n\) preserves the position in \(S_i\). The similar argument can be extended for the sets \(S_\beta\) and \(S_\gamma\) also.

\[\square\]

**Remark 2.3** Let \(k_0 = \left\lfloor \frac{2n-1}{3} \right\rfloor\) and \(l\) be an integer such that \(1 \leq l \leq k_0\). Let \(f\) be a minimal \(k\)-constrained total labeling of a path \(P_n\) and \(S_\alpha = \{\alpha, \alpha + 1, \alpha + 2, \ldots, \alpha + k_0 - 1\}\). Let \(S_l = \{3l - 2, 3l - 1, 3l\}\) and \(f(x) = \alpha + i\) for some \(x \in S_l\) then \(f(y) = \alpha + i + k\) implies \(y \in S_l\).

**Proof** After assigning the integers 1 to \(k_0\) one each for exactly one element of \(S_l\), for each \(l, 1 \leq l \leq k_0\), an unassigned element in the set containing the element labeled by 1 can be labeled by \(k + 1\). But no unassigned element of any other set can be labeled by \(k + 1\). Thus, if the label \(k + 1\) is not assigned to an element of the set whose one of the element is labeled by 1, then it should be excluded for the labeling of the elements of \(P_n\) and hence the number of isolated vertices required to make \(P_n\) a \(k\)-constrained graph will increase. Therefore, every minimal \(k\)-constrained total labeling should include label \(k + 1\) for an element of the set whose one of the element is labeled by 1. After including \(k + 1\), by continuing the same argument for \(k + 2, k + 3, \ldots, k + k_0\) one by one we can conclude that the label \(k + i\) (and then \(2k + i\)) can be labeled only for the element of the set whose one of the element is labeled by \(i\).

\[\square\]

**Remark 2.4** If \(1 \in f(S_l)\), then from the above Lemmas 2.1, 2.2 and Remark 2.3, it is clear that \(l, l + k, l + 2k \in f(S_l)\) for every \(l, 1 \leq l \leq k_0\), where \(k_0 = \left\lfloor \frac{2n-1}{3} \right\rfloor\).

**Lemma 2.5** Let \(S_l = \{3i - 2, 3i - 1, 3i\}\) and \(f\) be a minimal \(k\)-constrained total labeling of \(P_n\) such that \(f(x) = s\) for some \(x \in S_i\) for some \(i, 1 \leq i \leq k_0\), where \(k_0 = \left\lfloor \frac{2n-1}{3} \right\rfloor\). Then \(f(y) = s + 1\) implies \(y \in S_{l+1}\) or \(y \in S_{l-1}\) and hence by Lemma 2.2 we have \(|x - y| = 3\).

**Proof** Suppose the contrary that \(y \in S_j\) for some \(j\) where \(|j - i| > 1\) and \(1 \leq j \leq k_0\). Without loss of generality, we now assume that \(j > i + 1\) (otherwise relabel the set \(S_{k_0 - m}\) for each \(l, 1 \leq m \leq k_0\)). Now by repeated application of Lemma 2.1 we get the sequence of consecutive sets \(S_i, S_{i+1}, S_{i+2}, \ldots, S_j\) and the sequence of elements \(s = s_0, s_1 = s + 1, \ldots, s_{j-i} = s + 1\) where \(s_t \in S_{i+t}\) for each \(t, 0 \leq t \leq j\). As \(j > i + 1\), this sequence of elements (labels) is neither an increasing nor a decreasing sequence. So, there exists a positive integer \(l\) such that \(s_{l-1} < s_l \) and \(s_{l+1} < s_l\). Also, Remark 2.4 \(s_{l+k}, s_{l+2k} \in f(S_{l+i})\), \(s_{l+1+k}, s_{l+1+2k} \in f(S_{l+i+1})\) and \(s_{l-1+k}, s_{l-1+2k} \in f(S_{l+i-1})\). Let \(l_1 = 3(i + l) - 2, l_2 = 3(i + l) - 1, l_3 = 3(i + l)\). We now discuss the following 3! cases.

**Case 1** \(f(l_1) = s_l, f(l_2) = s_l + k, f(l_3) = s_l + 2k\).
In this case by Lemma 2.2 it follows that \( f(l_1 - 3) = s_{l_1 - 1}, f(l_2 - 3) = s_{l_1 - 1} + k, f(l_3 - 3) = s_{l_1 - 1} + 2k \) and \( f(l_1 + 3) = s_{l_1 + 1}, f(l_2 + 3) = s_{l_1 + 1} + k, f(l_3 + 3) = s_{l_1 + 1} + 2k \). So, \( |f(l_1 - 2) - f(l_1)| \geq k \Rightarrow |s_{l_1 - 1} + k - s_l| \geq k \Rightarrow |k - (s_l - s_{l_1 - 1})| \geq k \Rightarrow s_l - s_{l_1 - 1} \leq 0 \Rightarrow s_l \leq s_{l_1 - 1}, \) a contradiction.

**Case 2** \( f(l_1) = s_l, f(l_2) = s_l + 2k, f(l_3) = s_l + k \).

In this case by Lemma 2.2 it follows that \( f(l_1 - 3) = s_{l_1 - 1}, f(l_2 - 3) = s_{l_1 - 1} + 2k, f(l_3 - 3) = s_{l_1 - 1} + k \) and \( f(l_1 + 3) = s_{l_1 + 1}, f(l_2 + 3) = s_{l_1 + 1} + 2k, f(l_3 + 3) = s_{l_1 + 1} + k \). So, \( |f(l_1 - 1) - f(l_1)| \geq k \Rightarrow |(s_{l_1 - 1} + k) - (s_l + k)| \geq k \Rightarrow |k - (s_l - s_{l_1 - 1})| \geq k \Rightarrow s_l - s_{l_1 - 1} \leq 0 \Rightarrow s_l \leq s_{l_1 - 1}, \) a contradiction.

**Case 3** \( f(l_1) = s_l + k, f(l_2) = s_l, f(l_3) = s_l + 2k \).

In this case by Lemma 2.2 it follows that \( f(l_1 - 3) = s_{l_1 - 1} + k, f(l_2 - 3) = s_{l_1 - 1}, f(l_3 - 3) = s_{l_1 - 1} + 2k \) and \( f(l_1 + 3) = s_{l_1 + 1} + k, f(l_2 + 3) = s_{l_1 + 1}, f(l_3 + 3) = s_{l_1 + 1} + 2k \). So, \( |f(l_1 - 1) - f(l_1)| \geq k \Rightarrow |(s_{l_1 - 1} + k - (s_l + k))| \geq k \Rightarrow |k - (s_l - s_{l_1 - 1})| \geq k \Rightarrow s_l - s_{l_1 - 1} \leq 0 \Rightarrow s_l \leq s_{l_1 - 1}, \) a contradiction.

**Case 4** \( f(l_1) = s_l + 2k, f(l_2) = s_l, f(l_3) = s_l + k \).

In this case by Lemma 2.2 it follows that \( f(l_1 - 3) = s_{l_1 - 1} + 2k, f(l_2 - 3) = s_{l_1 - 1}, f(l_3 - 3) = s_{l_1 - 1} + k \) and \( f(l_1 + 3) = s_{l_1 + 1} + 2k, f(l_2 + 3) = s_{l_1 + 1}, f(l_3 + 3) = s_{l_1 + 1} + k \). So, \( |f(l_1 - 1) - f(l_2)| \geq k \Rightarrow |(s_{l_1 - 1} + k - s_l)| \geq k \Rightarrow |k - (s_l - s_{l_1 - 1})| \geq k \Rightarrow s_l - s_{l_1 - 1} \leq 0 \Rightarrow s_l \leq s_{l_1 - 1}, \) a contradiction.

**Case 5** \( f(l_1) = s_l + k, f(l_2) = s_l + 2k, f(l_3) = s_l \).

In this case by Lemma 2.2 it follows that \( f(l_1 - 3) = s_{l_1 - 1} + k, f(l_2 - 3) = s_{l_1 - 1} + 2k, f(l_3 - 3) = s_{l_1 - 1} \) and \( f(l_1 + 3) = s_{l_1 + 1} + k, f(l_2 + 3) = s_{l_1 + 1}, f(l_3 + 3) = s_{l_1 + 1} + k \). So, \( |f(l_3 + 1) - f(l_3)| \geq k \Rightarrow |(s_{l_1 + 1} + k - s_l)| \geq k \Rightarrow |k - (s_l - s_{l_1 + 1})| \geq k \Rightarrow s_l - s_{l_1 + 1} \leq 0 \Rightarrow s_l \leq s_{l_1 + 1}, \) a contradiction.

**Case 6** \( f(l_1) = s_l + 2k, f(l_2) = s_l + k, f(l_3) = s_l \).

In this case by Lemma 2.2 it follows that \( f(l_1 - 3) = s_{l_1 - 1} + 2k, f(l_2 - 3) = s_{l_1 - 1} + k, f(l_3 - 3) = s_{l_1 - 1} \) and \( f(l_1 + 3) = s_{l_1 + 1} + 2k, f(l_2 + 3) = s_{l_1 + 1} + k, f(l_3 + 3) = s_{l_1 + 1} \). So, \( |f(l_3 + 1) - f(l_2)| \geq k \Rightarrow |(s_{l_1 + 1} + 2k - (s_l + k))| \geq k \Rightarrow |k - (s_l - s_{l_1 + 1})| \geq k \Rightarrow s_l - s_{l_1 + 1} \leq 0 \Rightarrow s_l \leq s_{l_1 + 1}, \) a contradiction.

**Lemma 2.6** Let \( P_n \) be a path on \( n \) vertices and \( k_0 = \lfloor \frac{2n-1}{3} \rfloor \). Then \( t_k(P_n) \geq 2(k - k_0) - 1 \) whenever \( 2n \equiv 0 \pmod{3} \) and \( k > k_0 \).

**Proof** For \( 1 \leq l \leq k_0 \), let \( S_l = \{l_1, l_2, l_3\} \), where \( l_1 = 3l - 2, l_2 = 3l - 1, l_3 = 3l \). Let \( S_{k_0 + 1} = \{2n - 2, 2n - 1\} \) and \( T = \{1, 2, 3, ..., k_0\} \). Let \( f \) be a minimal \( k \)-constrained total labeling of \( P_n \), \( 2n \equiv 0 \pmod{3} \) and \( k > k_0 \), then by Lemma 2.1, we have \( |f(S_l) \cap T| = 1 \) for each \( i \) (i.e. exactly one element of \( S_l \) mapped to distinct element of \( T \) for each \( i, 1 \leq i \leq k_0 \)) and \( f(l_j) = m \in T \) for some \( j, 1 \leq j \leq 3 \), then for other element \( l_i \) of \( S_l, i \neq j \), we have \( |f(l_i) - f(l_j)| \geq k \) implies \( f(l_i) \geq k + m \). Thus \( f \) excludes the elements of the set \( T_1 = \{k_0 + 1, k_0 + 2, ..., k\} \) for the next assignments of the elements of \( S_l, i \neq k_0 + 1 \).

Let \( f(l_i) = t \) for some \( t \in T \), where \( l_i \in S_l \). Then for the minimum span \( f \), by Remark 2.3 \( f(l_j) = k + t \) for \( i \neq j \) and \( l_j \in S_l \).

Again by Lemma 2.3, we get \( |f(S_i) \cap T'\rangle = 1 \), for each \( i, 1 \leq i \leq k_0 \), where \( T' = \{k + 1, k + 1, k + 1, ..., k + k_0 - 1\} \).
Hence neither $P_S$ the second elements of each of the sets similar way we can argue that either $k$ $f$ $2$ in the first round of assignment and uses exactly one element of only for the element in $f$ and hence $f$ $f$. Lemma 2 of $f$ leaves at least $2(k - k_0)$ elements which are in the set $T_1 \cup T_2$.

In view of Lemma 2.2, there are only two possibilities for the assignments of elements of $S_{k_0+1}$ depending upon whether $f$ assigns an element of $T_1$ to an element of $S_{k_0+1}$ or not.

Let us now consider the first case. Let $x \in S_{k_0+1}$ such that $f(x) = t$ for some $t \in T_1$.

Claim $x = 2n - 1$

If not, $f(2n - 2) = t$, but then $f(2n - 3) \notin T \cup T_1$ and $f(2n - 4) \notin T \cup T_1$. Then by Lemma 2.2 $f(2n - 5) \in T \cup T_1$ and by Lemma 2.5 $f(2n - 5) = t - 1$. Then again as above $f(2n - 8) = t - 2$. Continuing this argument, we conclude that $f(1) = 1$ and $f(4) = 2$. But then, by above argument, we get $f(x) = k + 1$ and $f(x + 3) = k + 2$ for some $x \in S_1$ and $x \in \{2, 3\}$. So, $|f(x) - f(4)| = |k + 1 - 2| \geq k$ and $|4 - x| \leq 2$, a contradiction. Hence the claim.

By the above claim we get $f(2n - 1) \in T_1$. We now suppose that $f(2n - 2) \notin T_2$ (note that $f(2n - 2) \notin T \cup T_1$), then by above argument for the minimality of $f$ we have $f(2n - 2) = k + k_0 + 1$ and hence $f(1) = k + 1$ and $f(2) = 1$. So, by Lemma 2.5, $f(4) = k + 2$ and $f(5) = 2$. So, $f(3) \neq 2k + 1$ (Since $|f(3) - f(4)| = |2k + 1 - (k + 2)| \geq k$, which is inadmissible). This shows that $f$ includes either at most one element of $T_1 \cup T_2$ to label the elements of $S_{k_0+1}$ or leaves one more element namely $2k + 1$ to label the elements of $P_n$ (Since the label $2k + 1$ is possible only for the element in $S_1$. Thus $f$ leaves at least $2(k - k_0) - 1$ elements.

If the second case follows then the result is immediate because $f$ leaves $(k - k_0)$ elements in the first round of assignment and uses exactly one element of $T_2$ in the second round. □

Remark 2.7 In the above Lemma 2.6 if $2n \equiv 0 \pmod{3}$, then $t_k(P_n) \geq 2(k - k_0)$.

Proof If the hypothesis hold, then $S_{k_0+1} = \emptyset$ or $S_{k_0+1} = \{2n - 1\}$. In the first case, if $S_{k_0+1} = \emptyset$, then by the proof of the Lemma we see that any minimal $k$-constrained total labeling $f$ should leave exactly $2(k - k_0)$ integers for the labeling of the elements of the path $P_n$. In the second case when $S_{k_0+1} = 2n - 1$, by Lemma 2.5 $f(2n - 1) = k_0 + 1$ (we can assume that $f(1) \in f(S_1)$ because only other possibility by Lemma 2.5 is that the labeling of elements of $P_n$ is in the reverse order, in such a case relabel the sets $S_1$ as $S_{k_0+1}$). But then, again by Lemma 2.2 and Lemma 2.5 it forces to take $f(1) = 1$ and $f(4) = 2$ hence by Remark 2.4, $f(x) = k + 1$ only if $x = 2$ or $x = 3$. In either of the cases $|f(4) - f(x)| \geq k$, a contradiction. Hence neither $k_0 + 1$ nor $k + 1$ can be assigned. Further, if $k_0 + 1$ is not assigned, then in the similar way we can argue that either $k + k_0 + 1$ or $2k + 1$ can not be assigned while assigning the second elements of each of the sets $S_1, 1 \leq l \leq k_0$. Thus, in both the cases $f$ should leave at least $2(k - k_0)$ integers for the assignment of $P_n$, whenever $2n \equiv 0 \pmod{3}$. □

Theorem 2.8 Let $P_n$ be a path on $n$ vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then

$$t_k(P_n) = \begin{cases} 
0 & \text{if } k \leq k_0, \\
2(k - k_0) - 1 & \text{if } k > k_0 \text{ and } 2n \equiv 0(\text{mod } 3), \\
2(k - k_0) & \text{if } k > k_0 \text{ and } 2n \equiv 1 \text{ or } 2(\text{mod } 3). 
\end{cases}$$
Proof If \( k \leq k_0 \), then the result follows by Theorem 3.3 of [4]. Consider the case \( k > k_0 \).

Case i \( 2n \equiv 0 \) (mod 3)

By Lemma 2.6 we have \( t_k(P_n) \geq 2(k - k_0) - 1 \). Now, the function \( f : V(P_n) \cup E(P_n) \cup K_{2(k-k_0)-1} \rightarrow \{1, 2, \ldots, 2(n + k - k_0) - 2\} \) defined by \( f(1) = 2k + 1, f(2) = k + 1, f(3) = 1 \) and \( f(i) = f(i - 3) + 1 \) for all \( i, 4 \leq i \leq 2n - 3 \), \( f(2n - 2) = 2k + 1 + k_0, f(2n - 1) = k + 1 + k_0 \) and the vertices of \( K_{2(k-k_0)-1} \) to the remaining, is a Smarandachely \( k \)-constrained labeling of the graph \( P_n \cup K_{2(k-k_0)-1} \). Hence \( t_k(P_n) \leq 2(k - k_0) - 1 \).

Case ii \( 2n \not\equiv 0 \) (mod 3)

By Remark 2.7 we have \( t_k(P_n) \geq 2(k - k_0) \). On the other hand, the function \( f : V(P_n) \cup E(P_n) \cup K_{2(k-k_0)} \rightarrow \{1, 2, \ldots, 2(n + k - k_0) - 1\} \) defined by \( f(1) = 2k + 1, f(2) = k + 1, f(3) = 1 \), \( f(i) = f(i - 3) + 1 \) for all \( i, 4 \leq i \leq 2n - 1 \) and the vertices of \( K_{2(k-k_0)} \) to the remaining, is a Smarandachely \( k \)-constrained labeling of the graph \( P_n \cup K_{2(k-k_0)} \). Hence \( t_k(P_n) \leq 2(k - k_0) \).

Figure 1: A \( k \)-constrained total labeling of the path \( P_n \cup K_{2(k-k_0)} \), where \( 2n \equiv 2 \) (mod 3).

§3. \( k \)-Constrained Number of a Cycle

Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(C_n) = \{v_iv_{i+1} | 1 \leq i \leq n - 1\} \cup \{v_nv_1\} \). Due to the symmetry in \( C_n \), without loss of generality, we assume that the integer 1 is labeled to the vertex \( v_1 \) of \( C_n \). Define \( S_\alpha = \{\alpha_1, \alpha_2, \alpha_3\} \), for all \( \alpha \in \mathbb{Z}^+, 1 \leq \alpha \leq k_0 \), where \( k_0 = \left\lfloor \frac{2n}{3} \right\rfloor \) and \( \alpha_1 = v_{2n-1}, \alpha_2 = v_{2n+1}, \alpha_3 = v_{2n+2} \) for all odd \( \alpha \) and if \( \alpha \) is even, then and \( \alpha_1 = v_{2n-1}, \alpha_2 = v_{2n}, \alpha_3 = v_{2n+1} \).

**Case 1** \( 2n \equiv 0 \) (mod 3)

In this case set of elements (edges and vertices) of \( C_n \) is \( S_1 \cup S_2 \cup \cdots \cup S_{k_0} \cup S_{k_0+1} \), where \( S_{k_0+1} = \{v_{n+1}, \ldots, v_{n+1}\} \).

We now assume the contrary that \( t_k(C_n) < 2(k - k_0) \). Then there exists a minimal \( k \)-constrained labeling \( f \) such that span \( f \) is less that \( k_0 + 2k + 3 \) (since span \( f \) = number of vertices + edges + \( t_k(C_n) < 3(k_0 + 1) + 2(k - k_0) \)). Now our proof is based on the following observations.

**Observation 3.1** Let \( L_1 \) be the set of first possible consecutive integers (labels) that can be assigned for the elements of \( C_n \). Then exactly one element of each set \( S_\alpha, 1 \leq \alpha \leq k_0 + 1 \), can
receive one distinct label in \(L_1\) and for the minimum span all the labels in \(L_1\) to be assigned. Thus \(|L_1| = k_0 + 1\).

**Observation 3.2** The labels in \(L_1\) can be assigned only for the elements of \(S_\alpha\) in identical places (i.e. \(\alpha_1 \in S_\alpha\) receives \(f(\alpha_i) \in L_1\) and \(\beta_j \in S_\beta\) receives \(f(\beta_j) \in L_1\) if and only if \(i = j\) for all \(\alpha, \beta\)). In fact, since \(\alpha_1 = 1\), when \(\alpha = 1\), we get \(f(\beta_1) \in L_1\), where \(\beta = k_0 + 1\), hence \(f(\gamma_1) \in L_1\), where \(\gamma = k_0\), and so on \(\cdots\).

**Observation 3.3** The observation 3.2 holds for next labelings for the remaining unlabeled elements also.

**Observation 3.4** Since the smallest label in \(L_1\) is 1, by observation 3.1, it follows that the largest label in \(L_1\) is \(k_0 + 1\) and next minimum possible integer(label) in the set \(L_2\), consisting of consecutive integers used for the labeling of elements unassigned by the set \(L_1\), is \(k + 2\) (we observe that \(k + i\), for \(k_0 - k + 1 < i < 1\) can not be used for the labeling of any element in the set \(S_\alpha\), \(1 \leq \alpha \leq k_0 + 1\) (since an element of each of \(S_\alpha\) has already received a label \(x\) in \(L_1\), \(1 \leq x \leq k_0 + 1\) and \((k + i) - (x) = k + (i - x) < k\). Also if \(k + 1\) is assigned, then \(k + 1\) is assigned only to \(2^{nd}\) or \(3^{rd}\) element (viz \(\alpha_2\) or \(\alpha_3\), where \(\alpha = 1\)) of \(S_1\), but then difference of labels of first element of \(S_\alpha\) labeled by an integer in \(L_1\) (which is greater than 1) with \(k + 1\) differs by at most by \(k - 1\).

**Observation 3.5** By observation 3.4 it follows that the minimum integer label in \(L_2\) is \(k + 2\), so the maximum integer label is \(k + k_0 + 2\).

**Observation 3.6** Let \(L_3\) be the set of next consecutive integers which can be used for the labeling of the elements not assigned by \(L_1 \cup L_2\). Then, as span is less than \(k_0 + 2k + 3\), the maximum label in \(L_3\) is at most \(k_0 + 2k + 2\) and hence the minimum is at most \(2k + 2\).

We now suppose that \(f(\alpha_i) \in L_3\) and \(f(\alpha_i) = \min L_3\), for some \(\alpha, 1 \leq \alpha \leq k_0 + 1\). Then, as \(f(\alpha_i) = \min L_3\), \(f(\alpha_i) = 2k + j\) for some \(j \leq 2\). Further, as \(f(\alpha_i) \notin L_2\), we have \(k_0 + 2 - k \leq j \leq 2\). Combining these two we get \(k_0 + 2 - k \leq j \leq 2\).

**Subcase 1** \(i = 2\)

In this case \(f(\alpha_2) \in L_3\) and already \(f(\alpha_2) \in L_1\), so \(f(\alpha_3) \in L_2\) and hence \(f(\beta_1) \in L_2\) (by Observation 3.2), where \(\beta = \alpha - 1\) (or \(\beta = k_0 + 1\) if \(\alpha = 1\)). Thus, \(f(\beta_1) = k + l\) for some \(l, 2 \leq l \leq k + 2 + k_0\).

Now \(|f(\alpha_2) - f(\beta_3)| = |(2k + j) - (k + l)| = |k + (j - l)| \geq k| implies that \(j - l \geq 0\) hence \(j \geq l\). But \(j \leq 2 \leq l\) implies \(j = l = 2\). Therefore, \(f(\alpha_2) = 2k + 2\) and \(f(\beta_3) = k + l = k + 2 = \min L_2\).

In this case \(f(\alpha_3) \in L_2\) implies that \(f(\alpha_3) = k + m\), for some \(m > 2\). So, \(|f(\alpha_2) - f(\alpha_3)| = |(2k + 2) - (k + m)| = |k + (2 - m)| < k| as \(m > 2\), which is a contradiction.

**Subcase 2** \(i = 3\)

In this case \(f(\alpha_3) \in L_3\) and already \(f(\alpha_1) \in L_1\), so \(f(\alpha_2) \in L_2\) and hence \(f(\beta_2) \in L_2\) (by Observation 3.2), where \(\beta = \alpha - 1\) (or \(\beta = 1\) if \(\alpha = k_0 + 1\)). Thus, \(f(\beta_2) = k + l\) for some \(l, 2 \leq l \leq k + 2 + k_0\).

Now \(|f(\alpha_3) - f(\beta_2)| = |(2k + j) - (k + l)| = |k + (j - l)| \geq k| implies that \(j - l \geq 0\) hence
Since $f_{L_1}$ implies that $f(\alpha) = 2k + 2$ and $f(\beta_2) = k + l = k + 2 = \min L_2$.

In this case $f(\alpha_2) \in L_2$ implies that $f(\alpha_2) = k + m$, for some $m > 2$. So, $|f(\alpha_3) - f(\alpha_2)| = |(2k + 2) - (k + m)| = |k + (2 - m)| < k$ as $m > 2$, which is a contradiction.

Hence in either of the cases we get $t_k(C_n) \geq 2(k - k_0)$.

**Case 2** $2n \not\equiv 0 \pmod{3}$

Let $f$ be a minimal $k$-constrained total labeling of $C_n$. Let $L_1, L_2, L_3$ be the sets as defined as in Observations 3.1, 3.4 and 3.6 above. Let $L_4$ be the set of possible consecutive integers used for labeling the elements of $C_n$ which are not assigned by the set $L_1 \cup L_2 \cup L_3$.

We first take the case $2n \equiv 1 \pmod{3}$. If possible we now again assume the contrary that $t_k(C_n) < 3(k - k_0)$. Then it follows that span $f$ is less than $3k + 1$.

**Observation 3.7** Since minimum label in $L_1$ is 1 and $f$ is a minimal $k$-constrained labeling, we have $f(x) \geq k + 1$ for all $x$ such that $f(x) \in L_2$.

We have $f(\alpha_1) = 1$ for $\alpha = 1$. Let $\beta$ be the smallest index such that $f(\beta_1) \in L_1$ and $f(\gamma_1) \not\in L_1$, where $\gamma = \beta + 1$ (such index $\beta$ exists because $f(\alpha_1) = 1$ for $\alpha = 1$ and $\gamma$ exists because $2n \not\equiv 0 \pmod{3}$, the elements labeled by $L_1$ differ by its position by exactly multiples of 3 apart on either sides of the element labeled by 1). Now consider the set $S = \{\beta_2, \beta_3, \gamma_1\}$. None of the elements of $S$ can be labeled by any the label in $L_1$ and no two of them receive the label for a single set $L_i$, for any $i, 2 \leq i \leq 4$. Let $s_1, s_2, s_3$ be the elements of $S$ arranged accordingly $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$.

Since span $f \leq 3k$, we have $f(s_3) \leq 3k$, so $f(s_2) \leq 2k$ and hence $f(s_1) \leq k$, which is a contradiction (follows by Observation 3.7). Hence for any minimal $k$-constrained labeling $f$ we get $t_k(C_n) \geq 3(k - k_0)$ whenever $2n \equiv 1 \pmod{3}$.

We now take the case $2n \equiv 2 \pmod{3}$. If possible we now again assume the contrary that $t_k(C_n) < 3(k - k_0)$. Then it follows that span $f$ is less than or equal to $3k + 1$. The element of $C_n$ is the set $S_1 \cup S_2 \cup \cdots \cup S_{k_0} \cup S_{k_0+1}$, where $S_{k_0+1} = \{v_n, v_n, v_1\}$. We now claim that the label of the first element namely $\alpha_1$ of the set $S_\alpha$ is in the set $L_1$ for all $\alpha, 1 \leq \alpha \leq k_0$ if and only if $k_0 > 2$.

Suppose that $\alpha$ is the least positive index such that $f(\alpha_1) \not\in L_1$ and $1 < \alpha \leq k_0$. Then for all $\beta$ such that $1 \leq \beta < \alpha, f(\beta_1) \in L_1$. Let $\beta = \alpha - 1$. Consider the set $S = \{\beta_2, \beta_3, \alpha_1\}$. Let $s_1, s_2, s_3$ be the rearrangements of the elements in the set $S$ such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$ respectively.

Since $f(s_3) \in L_4$ and span $f$ is less than or equal to $3k + 1$ it follows that $f(s_3) \leq 3k + 1$ and hence $f(s_2) \leq 2k + 1, f(s_1) \leq k + 1$. But, the least element in $L_1$ is 1 implies that the least element in $L_2$ is greater or equal to $k + 1$, so $f(s_1) \geq k + 1$. Therefore, $f(s_1) = k + 1$, so that $f(s_2) = 2k + 1$ and $f(s_3) = 3k + 1$. There are two possible cases depending on $s_3 \in S_\alpha$ or not. Before considering these cases we make the the following observations.

**Observation 3.8** Since $f(\alpha_1) \in L_4$, we find $f(\alpha_1) = 3k + 1$ for any $\alpha > 1$. Suppose for any $\delta, \delta > \alpha$, if $f(\delta_1) \in L_1$, then for any $\gamma, \gamma > \delta$, we find $f(\gamma_1) \in L_1$. In fact, for $\gamma > \delta$, if $f(\gamma_1) \not\in L_1$ and $f(\eta_1) \in L_1$ for $\eta = \gamma - 1$, then sequence $s_1, s_2, s_3$ of the elements of the set $S = \{\eta_2, \eta_3, \gamma_1\}$
taken accordingly as \( f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4 \) as above, we get \( f(s_3) \leq 3k \) (since \( 3k + 1 \) is already assigned). Therefore, \( f(s_2) \leq 2k \) and hence \( f(s_1) \leq k \), which is impossible (since \( f(s_1) \notin L_1 \)).

This shows that if \( f(\delta_1) \in L_4 \), where \( \delta = \alpha + 1 \), we arrive at the situation that \( f(\eta_1) \in L_1 \), where \( \eta = k_0 \).

Now taking the set \( \{\eta_2, \eta_3, v_n\} \) and rearranging these elements as \( s_1, s_2, s_3 \) such that \( f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4 \), we get \( f(s_1) \leq k \) which is again a contradiction.

**Observation 3.9** Observation 3.8 shows that \( f(\delta_1) \notin L_1 \) for any \( \delta, \alpha < \delta \leq k_0 \).

**Observation 3.10** Starting from the vertex \( v_1 \), consider the sets \( \hat{S}_1 = \{v_1, v_1v_n, v_n\}, \hat{S}_2 = S_{k_0}, \hat{S}_3 = S_{k_0-1}, \ldots, \hat{S}_{k_0-\delta+2} = S_\delta \). By taking these sets, we arrive at the conclusion, as in Observation 3.8, that \( f(\delta_3) \in L_1 \) for every \( \delta > \alpha \).

We now continue the main proof for the first case \( s_3 \in S_\alpha \). In this case \( s_3 = \alpha_1 \), therefore \( s_1 \in S_\beta \). But \( f(s_3) \in L_4 \) implies that \( f(s_3) \leq 3k + 1 \), so \( f(s_2) \leq 2k + 1 \) and hence \( f(s_1) \leq k + 1 \). On the other hand \( f(\beta_1) \in L_1 \) implies that \( f(\beta_2) \) or \( f(\beta_3) \) is greater than or equal to \( k + 1 \) (since \( \min L_1 = 1 \)), that is, \( f(\beta_1) \geq k + 1 \). Thus, \( f(s_1) = k + 1 \). This yields \( f(\beta_1) = 1 \), so \( \beta = 1 \) and \( \alpha = 2 \). Also \( f(s_2) = 2k + 1 \) and \( f(s_3) = 3k + 1 \).

Let us now suppose that \( \alpha < k_0 \). Then there exists an index \( \delta \) such that \( \delta = \alpha + 1 \leq k_0 \).

If \( f(\beta_2) = 2k + 1, f(\beta_3) = k + 1 \), then \( f(\alpha_2) \geq 2k + 1 \) (since \( f(\beta_3) = k + 1 \)) and \( f(\alpha_2) \leq 2k + 1 \) (since \( f(\alpha_1) = 3k + 1 \)). So, \( f(\alpha_2) = 2k + 1 \) and hence \( f(\alpha_2) = f(\beta_2) \) which is not possible (since \( \alpha \neq \beta \)).

If \( f(\beta_2) = k + 1, f(\beta_3) = 2k + 1 \), then \( f(\alpha_2) \leq k + 1 \) implies \( f(\alpha_2) \in L_1 \) (since \( f(\alpha_2) \neq k + 1 = f(\beta_2) \)). Further by Observation 3.10, we have \( f(\delta_3) \in L_1 \). Consider the set \( \{\alpha_3, \delta_1, \delta_2\} \) (we note that none of the elements of this set is labeled by the set \( L_1 \)) and let \( s_1, s_2, s_3 \) be the elements of this set taken in order such that \( f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4 \). Since \( 3k + 1 \) is already assigned we get \( f(s_3) \leq 3k \) and hence as above \( f(s_1) \leq k \), which is a contradiction to the fact \( f(s_1) \notin L_1 \).

We now continue the main proof for the second case \( s_3 \notin S_\alpha \). In this case \( s_3 \in S_\beta \). Now by assumption we have \( f(\alpha_3) \in L_1 \) and \( k + 1 \) is already labeled for an element of \( S_\beta = S_1 \), therefore \( f(\alpha_1) = 2k + 1 \). Now by Observation 3.10, \( f(\delta_3) \in L_1 \), where \( \delta = \alpha + 1 \). If \( f(\alpha_2) \in L_1 \), then by taking the set \( \{\alpha_3, \delta_1, \delta_2\} \) and arranging as above we can show that one of these elements must be labeled by an element of the set \( L_4 \) and hence that label should be at most \( 3k \), so the smallest label of the element of the set is less than or equal \( k \), a contradiction to the fact that the smallest label is not in \( L_1 \). Thus, \( f(\alpha_2) \notin L_1 \).

If \( f(\delta_3) = 3k + 1 \), then \( f(\alpha_2) \in L_2 \), and hence \( f(\alpha_2) \geq k + 2 \), which is not possible because \( f(\alpha_1) = 2k + 1 \). Therefore, \( f(\beta_2) = 3k + 1 \) and \( f(\beta_3) = k + 1 \). But then, only possibility is that \( f(\alpha_2) \in L_4 \) implies that \( f(\alpha_2) \leq 3k \), which is impossible because \( f(\alpha_1) = 2k + 1 \). Hence the claim.

By the above claim we have either first element of all the sets \( S_1, S_2, \ldots, S_{k_0} \) are labeled by the elements of the set \( L_1 \) or the graph is the cycle \( C_4 \). For the graph \( C_4 \), it is easy to observe that no three consecutive integers can be used for the labeling and hence each of the sets \( L_1, L_2, L_3 \) and \( L_4 \) should have at most two elements. Thus, \( \text{span } f \geq 3k + 2 \). The equality
holds by the following Figure 2.

\[ \begin{array}{c}
1 & k+1 & 2k+2 \\
2k+1 & 3k+2 & k+2 \\
3k+1 & & \\
\end{array} \]

Figure 2: A k-constrained total labeling of the graph \(C_4 \cup \overline{K}_{3k-6}\)

If the graph is not \(C_4\), then consider the set \(T = \{v_{n-1}, v_{n-1}v_n, v_nv_1\}\). Since \(f(v_{n-2}v_{n-1}) \in L_1\) (follows by Observation 3.10) and \(f(v_1) = 1 \in L_1\) (follows by the assumption) none of the elements of the set \(T\) is labeled by the set \(L_1\) and exactly two elements namely \(v_{n-1}\) and \(v_nv_1\) are labeled by same set.

If \(f(v_{n-1})\) and \(f(v_nv_1)\) are in \(L_2\), then either \(f(v_{n-1}v_n)\) and \(f(v_n)\) is in \(L_4\). Suppose \(f(v_{n-1}v_n)\) (similarly \(f(v_n)\) \(\in L_4\)), then \(f(v_n) \in L_3\) (\(f(v_{n-1}v_n) \in L_3\)), so \(f(v_{n-1}v_n) \leq 3k+1\) and hence \(f(v_n) \leq 2k+1\). Therefore both \(f(v_{n-1})\) and \(f(v_nv_1)\) must be less than or equal to \(k+1\), which is not possible because minimum of \(L_2\) is \(k+1\).

If \(f(v_{n-1})\) and \(f(v_nv_1)\) are in \(L_3\), then \(f(v_n) \in L_4\) (or \(f(v_{n-1}v_n) \in L_4\)) so \(f(v_nv_1) \leq 2k+1\) and \(f(v_{n-1}) \leq 2k+1\) (since \(f(v_n) \leq 3k+1\)). Therefore, at least one of \(f(v_nv_1)\) or \(f(v_{n-1})\) is less than or equal to \(2k\), which yields that \(f(v_{n-1}v_n) \leq k\) (\(f(v_n) \leq k\)). Thus, either \(f(v_{n-1}v_n)\) or \(f(v_nv_1)\) are in \(L_1\), a contradiction.

If \(f(v_{n-1})\) and \(f(v_nv_1)\) are in \(L_4\), then at least one of them must be less than \(3k+1\). Hence either \(f(v_n)\) or \(f(v_{n-1}v_1)\) is less than or equal to \(k\) (as above), which is again a contradiction.

Thus, we conclude

**Lemma 3.11** Let \(C_n\) be a cycle on \(n\) vertices and \(k_0 = \lfloor \frac{2n-1}{3} \rfloor\). Then

\[
t_k(C_n) \geq \begin{cases} 
0 & \text{if } k \leq k_0, \\
2(k-k_0) & \text{if } k > k_0 \text{ and } 2n \equiv 0 \text{ (mod 3)}, \\
3(k-k_0) & \text{if } k > k_0 \text{ and } 2n \equiv 1 \text{ or } 2 \text{ (mod 3)}. 
\end{cases}
\]

Now to prove the reverse inequality, designate the vertex \(v_i\) of \(C_n\) as \(2i-1\) and the edge \(v_iv_{j+1}\) as \(2j, v_nv_1\) as \(2n\). For each \(i, 1 \leq i \leq n\) and \(1 \leq j \leq n-1\) and for the case \(2n \equiv 0\) (mod 3), define a function \(f: V(C_n) \cup E(C_n) \cup V(\overline{K}_{2(k-k_0)}) \rightarrow \{1, 2, 3, \ldots, 2k+k_0+3\}\) by \(f(1) = 1, f(2) = k+2, f(3) = 2k+3, f(i) = f(i-3) + 1, 4 \leq i \leq 2n\) and the vertices of \(\overline{K}_{2(k-k_0)}\) to the remaining.
The function $f$ serves as a Smarandachely $k$-constrained labeling of the graph $C_n \cup \overline{K}_{2(k-k_0)}$. Hence $t_k(C_n) \leq 2(k - k_0)$.

For the case $2n \equiv 1 \pmod{3}$, define a function $f : V(C_n) \cup V(C_n) \cup V(\overline{K}_{3(k-k_0)}) \to \{1, 2, 3, \ldots, 3k + 1\}$ by $f(1) = 1, f(2) = 2k + 2, f(3) = k + 2, f(i) = f(i - 3) + 1$ for $4 \leq i \leq 2n - 4, f(2n - 3) = k_0, f(2n - 2) = 3k + 1, f(2n - 1) = 2k + 1, f(2n) = k + 1$ and the vertices of $\overline{K}_{3(k-k_0)}$ to the remaining.

The function $f$ serves as a Smarandachely $k$-constrained labeling of the graph $C_n \cup \overline{K}_{3(k-k_0)}$. Hence $t_k(C_n) \leq 3(k - k_0)$.

For the case $2n \equiv 2 \pmod{3}$, define a function $f : V(C_n) \cup V(C_n) \cup V(\overline{K}_{3(k-k_0)}) \to \{1, 2, 3, \ldots, 3k + 2\}$ by $f(1) = 1, f(2) = k + 2, f(3) = 2k + 3, f(i) = f(i - 3) + 1$ for $4 \leq i \leq 2n - 6, f(2n - 5) = 3k + 1, f(2n - 4) = k_0, f(2n - 3) = 2k + 1, f(2n - 2) = 3k + 2, f(2n - 1) = k + 1, f(2n) = 2k + 2$ the vertices of $\overline{K}_{3(k-k_0)}$ to the remaining.

The function $f$ serves as a Smarandachely $k$-constrained labeling of the graph $C_n \cup \overline{K}_{3(k-k_0)}$. Hence $t_k(C_n) \leq 3(k - k_0)$.

Hence, in view of Lemma 3.11, we get

**Theorem 3.12** Let $C_n$ be a cycle on $n$ vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then

$$
t_k(C_n) = \begin{cases} 
0 & \text{if } k \leq k_0, \\
2(k - k_0) & \text{if } k > k_0 \text{ and } 2n \equiv 0 \pmod{3}, \\
3(k - k_0) & \text{if } k > k_0 \text{ and } 2n \equiv 1 \text{ or } 2 \pmod{3}.
\end{cases}
$$

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