THE SMARANDACHE MULTIPLICATIVE FUNCTION

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Abstract
For any positive integer $n$, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b)), (a, b) = 1$. Now for any prime $p$ and any positive integer $\alpha$, we take $f(p^\alpha) = \alpha p$. It is clear that $f(n)$ is a Smarandache multiplicative function. In this paper, we study the mean value properties of $f(n)$, and give an interesting mean value formula for it.

Keywords: Smarandache multiplicative function; Mean Value; Asymptotic formula.

§1 Introduction and results

For any positive integer $n$, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b)), (a, b) = 1$. Now for any prime $p$ and any positive integer $\alpha$, we take $f(p^\alpha) = \alpha p$. It is clear that $f(n)$ is a new Smarandache multiplicative function, and if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime powers factorization of $n$, then

$$f(n) = \max_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\} = \max_{1 \leq i \leq k} \{\alpha_ip_i\}. \tag{1}$$

About the arithmetical properties of $f(n)$, it seems that none had studied it before. This function is very important, because it has many similar properties with the Smarandache function $S(n)$ (see reference [1][2]). The main purpose of this paper is to study the mean value properties of $f(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} f(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2 Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. For convenience, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime powers factorization of $n$, and $P(n)$ be the greatest prime factor of $n$, that is, $P(n) = \max_{1 \leq i \leq k} \{p_i\}$. Then we have
Lemma. For any positive integer \(n\), if there exists \(P(n)\) such that \(P(n) > \sqrt{n}\), then we have the identity

\[ f(n) = P(n). \]

Proof. From the definition of \(P(n)\) and the condition \(P(n) > \sqrt{n}\), we get

\[ f(P(n)) = P(n). \] (2)

For other prime divisors \(p_i\) of \(n\) \((1 \leq i \leq k\) and \(p_i \neq P(n)\)), we have

\[ f(p_i^{\alpha_i}) = \alpha_i p_i. \]

Now we will debate the upper bound of \(f(p_i^{\alpha_i})\) in three cases:

(I) If \(\alpha_i = 1\), then \(f(p_i) = p_i \leq \sqrt{n}\).

(II) If \(\alpha_i = 2\), then \(f(p_i^2) = 2p_i \leq 2 \cdot n^{1/2} \leq \sqrt{n}\).

(III) If \(\alpha_i \geq 3\), then \(f(p_i^{\alpha_i}) = \alpha_i \cdot p_i \leq \alpha_i \cdot n^{1/2} \leq n^{1/2} \cdot \frac{\ln n}{\ln p_i} \leq \sqrt{n},\)

where we use the fact that \(\alpha \leq \frac{\ln n}{\ln p_i}\) if \(p_i^\alpha | n\).

Combining (I)-(III), we can easily obtain

\[ f(p_i^{\alpha_i}) \leq \sqrt{n}. \] (3)

From (2) and (3), we deduce that

\[ f(n) = \max_{1 \leq i \leq k} \{ f(p_i^{\alpha_i}) \} = f(P(n)) = P(n). \]

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of the theorem. First we define two sets \(A\) and \(B\) as following:

\[ A = \{ n | n \leq x, P(n) \leq \sqrt{n} \}, \quad B = \{ n | n \leq x, P(n) > \sqrt{n} \}. \]

Using the Euler summation formula (see reference [3]), we may get

\[
\sum_{n \in A} f(n) \ll \sum_{n \leq x} \sqrt{n} \ln n \\
= \int_1^x \sqrt{t} \ln t \, dt + \int_1^x (t - [t])(\sqrt{t} \ln t)\prime \, dt + \sqrt{x} \ln x (x - [x]) \\
\ll x^{3/2} \ln x.
\]

Similarly, from the Abel’s identity we also have

\[
\sum_{n \in B} f(n) = \sum_{n \leq x} P(n) = \sum_{n \leq \sqrt{x}} \sum_{n \leq \sqrt{x} p \leq \frac{x}{n}} p \\
= \sum_{n \leq \sqrt{x}} \sum_{\sqrt{x} \leq p \leq \frac{x}{n}} p + O \left( \sum_{n \leq \sqrt{x}} \sum_{\sqrt{x} \leq p \leq \frac{x}{n}} \sqrt{x} \right) \\
= \sum_{n \leq \sqrt{x}} \left( \frac{x}{n} \pi \left( \frac{x}{n} \right) - \sqrt{x} \pi (\sqrt{x}) - \int_\sqrt{x}^{\sqrt{x}} \pi(s) \, ds \right) + O \left( x^{3/2} \ln x \right),
\]

(5)
where $\pi(x)$ denotes all the numbers of prime which is not exceeding $x$. Note that

$$\pi(x) = \frac{x}{\ln x} + O \left( \frac{x}{\ln^2 x} \right),$$

from (5) we have

$$\sum_{\sqrt{x} \leq n \leq x} \frac{x^2}{n^2 \ln x/n} = \frac{x}{n} \left( \frac{x}{n} \right) - \sqrt{x} \pi(\sqrt{x}) - \int_{\sqrt{x}}^{x} \pi(s) ds$$

$$= \frac{1}{2} \cdot \frac{x^2}{n^2 \ln x/n} - \frac{1}{2} \cdot \frac{x}{\ln \sqrt{x}} + O \left( \frac{x^2}{n^2 \ln^2 x/n} \right)$$

$$+ O \left( \frac{x}{\ln^2 \sqrt{x}} \right) + O \left( \frac{x^2}{n^2 \ln^2 x/n} - \frac{x}{\ln^2 \sqrt{x}} \right). \quad (6)$$

Hence

$$\sum_{n \leq \sqrt{x}} \frac{x^2}{n^2 \ln x/n} = \sum_{n \leq \sqrt{x}} \frac{x^2}{n^2 \ln x/n} + O \left( \sum_{n^2 \leq x \leq \sqrt{x}} \frac{x^2}{n^2 \ln x/n} \right)$$

$$= \frac{\pi^2}{6} \cdot \frac{x^2}{\ln x} + O \left( \frac{x^2}{\ln^2 x} \right), \quad (7)$$

and

$$\sum_{n \leq \sqrt{x}} \frac{x^2}{n^2 \ln^2 x/n} = O \left( \frac{x^2}{\ln^2 x} \right). \quad (8)$$

From (4), (5), (6), (7) and (8), we may immediately deduce that

$$\sum_{n \leq x} f(n) = \sum_{n \in A} f(n) + \sum_{n \in B} f(n)$$

$$= \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O \left( \frac{x^2}{\ln^2 x} \right).$$

This completes the proof of the theorem.

Note. If we use the asymptotic formula

$$\pi(x) = \frac{x}{\ln x} + \frac{c_1 x}{\ln^2 x} + \cdots + \frac{c_m x}{\ln^n x} + O \left( \frac{x}{\ln^{n+1} x} \right)$$

to substitute

$$\pi(x) = \frac{x}{\ln x} + O \left( \frac{x}{\ln^2 x} \right)$$

in (5) and (6), we can get a more accurate asymptotic formula for $\sum_{n \leq x} f(n)$. 

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References