On the property of the Smarandache-Riemann zeta sequence

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Abstract In this paper, some elementary methods are used to study the property of the Smarandache-Riemann zeta sequence and obtain a general result.

Keywords Riemann zeta function, Smarandache-Riemann zeta sequence, positive integer.

§1. Introduction and result

For any complex number $s$, let

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

be the Riemann zeta function. For any positive integer $n$, let $T_n$ be a positive real number such that

$$\zeta(2n) = \frac{\pi^{2n}}{T_n},$$

(1)

where $\pi$ is ratio of the circumference of a circle to its diameter. Then the sequence $T = \{T_n\}_{n=1}^{\infty}$ is called the Smarandache-Riemann zeta sequence. About the elementary properties of the Smarandache-Riemann zeta sequence, some scholars have studied it, and got some useful results. For example, in [2], Murthy believed that $T_n$ is a sequence of integers. Meanwhile, he proposed the following conjecture:

**Conjecture.** No two terms of $T_n$ are relatively prime.

In [3], Le Maohua proved some interesting results. That is, if

$$ord(2, (2n)!) < 2n - 2,$$

where $ord(2, (2n)!)$ denotes the order of prime 2 in $(2n)!$, then $T_n$ is not an integer, and finally he defies Murthy’s conjecture.

In reference [4], Li Jie proved that for any positive integer $n \geq 1$, we have the identity

$$ord(2, (2n)!) = a_2(2n) \equiv \sum_{i=1}^{\infty} \left\lfloor \frac{2n}{2^i} \right\rfloor = 2n - a(2n, 2),$$

where $[x]$ denotes the greatest integer not exceeding $x$.

So if $2n - a(2n, 2) < 2n - 2$, or $a(2n, 2) \geq 3$, then $T_n$ is not an integer.
In fact, there exist infinite positive integers \( n \) such that \( a(2n, 2) \geq 3 \), and \( T_n \) is not an integer. From this, we know that Murthy’s conjecture is not correct, because there exist infinite positive integers \( n \) such that \( T_n \) is not an integer.

In this paper, we use the elementary methods to study another property of the Smarandache-Riemann zeta sequence, and give a general result for it. That is, we shall prove the following conclusion:

**Theorem.** If \( T_n \) are positive integers, then 3 divides \( T_n \), more generally, if \( n = 2^k \), then 5 divides \( T_n \); If \( n = 3^k \), then 7 divides \( T_n \), where \( k \neq 0 \) is an integer.

So from this Theorem we may immediately get the following

**Corollary.** For any positive integers \( m \) and \( n (m \neq n) \), if \( T_m \) and \( T_n \) are integers, then

\[
(T_m, T_n) \geq 3, \quad (T_{2m}, T_{2n}) \geq 15, \quad (T_{3m}, T_{3n}) \geq 21.
\]

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we need two simple Lemmas which we state as follows:

**Lemma 1.** If \( n \) is a positive integer, then we have

\[
\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad (2)
\]

where \( B_{2n} \) is the Bernoulli number.

**Proof.** See reference [1].

**Lemma 2.** For any positive integer \( n \), we have

\[
B_{2n} = I_n - \sum_{p-1|2n} \frac{1}{p}, \quad (3)
\]

where \( I_n \) is an integer and the sum is over all primes \( p \) such that \( p - 1 \) divides \( 2n \).

**Proof.** See reference [3].

**Lemma 3.** For any positive integer \( n \), we have

\[
T_n = \frac{(2n)!b_n}{2^{2n-1}a_n}, \quad (4)
\]

where \( a_n \) and \( b_n \) are coprime positive integers satisfying \( 2||b_n, 3|b_n, n \geq 1 \).

**Proof.** It is a fact that

\[
\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n)!} \cdot B_{2n}, \quad n \geq 1,
\]

where

\[
B_{2n} = (-1)^{n-1} \frac{a_n}{b_n}, \quad n \geq 1. \quad (6)
\]

Using (1), (5) and (6), we get (4).

Now we use above Lemmas to complete the proof of our theorem.
For any positive integer \( n \), from (4) we can directly obtain that if \( T_n \) is an integer, then 3 divides \( T_n \), since \((a_n, b_n) = 1\).

From (1), (2) and (3) we have the following equality

\[
\zeta(2n) = \frac{\pi^{2n}}{T_n} = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} \left(I_n - \sum_{p \mid 2n} \frac{1}{p}\right),
\]

Let

\[
\prod_{p \mid 2n} p = p_1 p_2 \cdots p_s,
\]

where \( p_i (1 \leq i \leq s) \) is a prime number, and \( p_1 < p_2 \cdots < p_s \).

Then from the above, we have

\[
T_n = \frac{(-1)^{n+1} \cdot \pi^{2n}}{2(2n)!} \cdot \left(I_n - \sum_{p \mid 2n} \frac{1}{p}\right) = \frac{(-1)^{n+1} \cdot (2n)!}{2^{2n-1} \cdot (2n)! \cdot \prod_{p \mid 2n} p} \cdot \left(I_n - \sum_{p \mid 2n} \frac{1}{p}\right) = \frac{(-1)^{n+1} \cdot (2n)! \cdot p_1 p_2 \cdots p_s}{2^{2n-1} \cdot (I_n \cdot p_1 p_2 \cdots p_s - p_2 p_3 \cdots p_s + p_1 p_3 \cdots p_s - \cdots - p_1 p_2 \cdots p_{s-1})}
\]

Then we find that if \( p_i \mid p_1 p_2 \cdots p_s \), \( 1 \leq i \leq s \), but

\[
p_i \nmid (I_n \cdot p_1 p_2 \cdots p_s - p_2 p_3 \cdots p_s + p_1 p_3 \cdots p_s - \cdots - p_1 p_2 \cdots p_{s-1})
\]

So we can easily deduce that if \( T_n \) are integers, when \( n = 2k \), 5 can divide \( T_n \); While \( n = 3k \), then 7 can divide \( T_n \).

This completes the proof of Theorem.

\section*{References}


