SOME EXPRESSIONS OF THE SMARANDACHE PRIME FUNCTION

Sebastian Martin Ruiz
Avda. de Regla 43, Chipiona 11550, Spain

Abstract
The main purpose of this paper is using elementary arithmetical functions to give some expressions of the Smarandache Prime Function $P(n)$.

In this article we gave some expressions of the Smarandache Prime Function $P(n)$ (see reference [1]), using elementary arithmetical functions. The Smarandache Prime Function is the complementary of the Prime Characteristic Function:

$$P(n) = \begin{cases} 0 & \text{if } n \text{ is a prime,} \\ 1 & \text{if } n \text{ is a composite.} \end{cases}$$

Expression 1.

$$P(n) = 1 \left\lfloor \frac{lcm(1, 2, \cdots, n)}{n \cdot \gcd(1, 2, \cdots, n - 1)} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function (see reference [2]).

Proof. We consider three cases:

Case 1: If $n = p$ with $p$ prime, then we have

$$lcm(1, 2, \cdots, p) = lcm(lcm(1, 2, \cdots, p - 1), p) = p \cdot lcm(1, 2, \cdots, p - 1)$$

Therefore we have: $P(n) = 0$.

Case 2: If $n = p^\alpha$ with $p$ is prime and $\alpha$ is a positive integer greater than one, we may have

$$\left\lfloor \frac{lcm(1, 2, \cdots, n)}{n \cdot lcm(1, 2, \cdots, n - 1)} \right\rfloor$$

$$= \left\lfloor \frac{lcm(1, 2, \cdots, n^\alpha)}{n \cdot lcm(1, 2, \cdots, n^\alpha - 1)} \right\rfloor$$

$$= \left\lfloor \frac{lcm(lcm(1, 2, \cdots, n^{\alpha - 1}, \cdots, n^\alpha - 1), p^\alpha)}{n \cdot lcm(1, 2, \cdots, n^\alpha - 1)} \right\rfloor$$
\[
\begin{align*}
&= \left\lfloor \frac{p \cdot \text{lcm}(1, 2, \cdots, p^{a-1}, \cdots, p^a - 1)}{n \cdot (1, 2, \cdots, p^a - 1)} \right\rfloor \\
&= \left\lfloor \frac{p}{n} \right\rfloor = 0.
\end{align*}
\]

So we have: \( P(n) = 1. \)

Case 3: If \( n = a \cdot b \) with \( \gcd(a, b) = 1 \) and \( a, b > 1 \). We can suppose \( a < b \), then we have

\[
\begin{align*}
lcm(1, 2, \cdots, a, \cdots, b, \cdots, n) \\
&= lcm(1, 2, \cdots, a, \cdots, b, \cdots, n - 1, a \cdot b) \\
&= lcm(1, 2, \cdots, a, \cdots, b, \cdots, n - 1)
\end{align*}
\]

and therefore we have:

\[
\begin{align*}
P(n) &= 1 - \left\lfloor \frac{lcm(1, 2, \cdots, n)}{n \cdot lcm(1, 2, \cdots, n - 1)} \right\rfloor \\
&= 1 - \left\lfloor \frac{1}{n} \right\rfloor = 1 - 0 = 1
\end{align*}
\]

With this the expression one is proven.

**Expression 2.** [3],[4]

\[
P(n) = -\left\lfloor 2 - \sum_{i=1}^{n} \frac{n}{i} - \left\lfloor \frac{n-1}{i} \right\rfloor \right\rfloor
\]

**Proof.** We consider \( d(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \) is the number of divisors of \( n \) because:

\[
\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = \begin{cases} 1 & \text{if } i \text{ divides } n, \\ 0 & \text{if } i \text{ not divide } n. \end{cases}
\]

If \( n = p \) prime we have \( d(n) = 2 \) and therefore \( P(n) = 0. \)

If \( n \) is composite we have \( d(n) > 2 \) and therefore:

\[
-1 < \frac{2 - d(n)}{n} < 0 \implies P(n) = 1.
\]

**Expression 3.**

We can also prove the following expression:
Some Expressions of the Smarandache Prime Function

\[ P(n) = 1 - \left\lfloor \frac{1}{n} \cdot \text{GCD} \left( \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n-1} \right) \right\rfloor, \]

where \( \binom{n}{i} \) is the binomial coefficient.

Can the reader prove this last expression?

References


