Some identities on $k$-power complement

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Abstract The main purpose of this paper is to calculate the value of the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^n(n)},$$

where $a_k(n)$ is the $k$-power complement number of any positive number $n$, and $\alpha, \beta$ are two complex numbers with $\text{Re}(\alpha) \geq 1$, $\text{Re}(\beta) \geq 1$. Several interesting identities are given.

Keywords $k$-power complement number, identities, Riemann zeta-function.

§1. Introduction

For any given natural number $k \geq 2$ and any positive integer $n$, we call $a_k(n)$ as a $k$-power complement number if $a_k(n)$ denotes the smallest positive integer such that $n \cdot a_k(n)$ is a perfect $k$-power. Especially, we call $a_2(n), a_3(n), a_4(n)$ as the square complement number, cubic complement number, quartic complement number respectively. In reference [1], Professor F.Smarandache asked us to study the properties of the $k$-power complement number sequence. About this problem, there are many authors had studied it, and obtained many results. For example, in reference [2], Professor Wenpeng Zhang calculated the value of the series

$$\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_k(n))^s},$$

where $s$ is a complex number with $\text{Re}(\alpha) \geq 1$, $k=2, 3, 4$. Maohua Le [3] discussed the convergence of the series

$$s_1 = \sum_{n=1}^{+\infty} \frac{1}{a_2^n(n)},$$

and

$$s_2 = \sum_{n=2}^{+\infty} \frac{(-1)^n}{a_2(n)},$$

where $m \leq 1$ is a positive number, and proved that they are both divergence.

But about the properties of the $k$-power complement number, we still know very little at present. This paper, as a note of [2], we shall give a general calculate formula for

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^n(n)},$$
That is, we shall prove the following:

**Theorem 1.** For any complex numbers \( \alpha, \beta \) with \( \Re(\alpha) \geq 1, \Re(\beta) \geq 1 \), we have

\[
\sum_{n=1}^{+\infty} \frac{1}{n^\alpha \cdot a_k^n(n)} = \zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^{\alpha+(k-1)\beta} - 1} \right),
\]

where \( \zeta(\alpha) \) is the Riemann zeta-function, \( \prod_p \) denotes the product over all prime \( p \).

**Theorem 2.** For any complex numbers \( \alpha, \beta \) with \( \Re(\alpha) \geq 1, \Re(\beta) \geq 1 \), we have

\[
\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^n(n)} = \left( 1 - \frac{2(2^{k\alpha} - 1)(2^{\alpha+(k+1)\beta} - 1)}{2^{k+1} + (k-1)\beta - 2^{\alpha+(k-1)\beta}} \right) \zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^{\alpha+(k-1)\beta} - 1} \right).
\]

Note that \( \zeta(2) = \frac{\pi^2}{6} \), \( \zeta(4) = \frac{\pi^4}{90} \) and \( \zeta(8) = \frac{\pi^8}{9450} \). From our Theorems we may immediately obtain the following two corollaries:

**Corollary 1.** Taking \( \alpha = \beta, k = 2 \) in above Theorems, then we have

\[
\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^\alpha} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)}; \\
\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^\alpha \cdot 2|n|} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)} \cdot \frac{4^\alpha - 1}{4^\alpha + 1}; \\
\sum_{n=1}^{+\infty} \frac{(-1)^n}{(n \cdot a_2(n))^\alpha} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)} \cdot \frac{3 - 4^\alpha}{1 + 4^\alpha}.
\]

**Corollary 2.** Taking \( \alpha = \beta = 1, 2, k = 2 \) in Corollary 1, we have

\[
\sum_{n=1}^{+\infty} \frac{1}{n \cdot a_2(n)} = \frac{5}{2}; \\
\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^2} = \frac{7}{6}; \\
\sum_{n=1}^{+\infty} \frac{1}{2|n| \cdot a_2(n)} = \frac{3}{2}; \\
\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^2} = \frac{35}{34}; \\
\sum_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot a_2(n)} = -\frac{1}{2}; \\
\sum_{n=1}^{+\infty} \frac{(-1)^n}{(n \cdot a_2(n))^2} = \frac{91}{102}.
\]
§2. Proof of the theorem

In this section, we will complete the proof of the theorems. For any positive integer \( n \), we can write it as \( n = m^k \cdot l \), where \( l \) is a \( k \)-free number, then from the definition of \( a_k(n) \) we have

\[
\frac{1}{n^\alpha \cdot a_k^\alpha(n)} = \frac{1}{\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \sum_{d^k | l} \mu(d) m^{\alpha_1 \alpha} 1^{(k-1)\beta} l} \]

\[
\frac{1}{\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \sum_{d^k | l} \mu(d) m^{\alpha_1 \alpha} 1^{(k-1)\beta} l} = \zeta(k\alpha) \sum_{l=1}^{\infty} \sum_{l=1}^{\infty} \sum_{d^k | l} \mu(d) m^{\alpha_1 \alpha} 1^{(k-1)\beta} l
\]

\[
\zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} + \frac{1}{p^{2\alpha + (k-1)\beta}} + \cdots + \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]

\[
\zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} - \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]

\[
\zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} - \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]

where \( \mu(n) \) denotes the Möbius function. This completes the proof of Theorem 1.

Now we come to prove Theorem 2. First we shall prove the following identity

\[
\sum_{m=1}^{\infty} \frac{1}{n^\alpha \cdot a_k^\alpha(n)} = \sum_{m=1}^{\infty} \frac{1}{m^{\alpha_1 \alpha} 1^{(k-1)\beta} l}
\]

\[
\sum_{m=1}^{\infty} \frac{1}{m^{\alpha_1 \alpha} 1^{(k-1)\beta} l} \sum_{d^k | l} \mu(d) m^{\alpha_1 \alpha} 1^{(k-1)\beta} l
\]

\[
\zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} - \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]

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\zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} - \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]

\[
\zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} - \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]

Then use this identity and Theorem 1 we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^\alpha(n)} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha \cdot a_k^\alpha(n)} - \sum_{n=1}^{\infty} \frac{1}{2n^\alpha \cdot a_k^\alpha(n)}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^\alpha(n)} = \left( 1 - \frac{2(2k\alpha - 1)(2\alpha + (k-1)\beta - 1)}{2(k+1)\alpha + (k-1)\beta - 2(k-1)^2\beta - \alpha} \right) \zeta(k\alpha) \prod_p \left( 1 + \frac{1}{p^\alpha (k-1)\beta} - \frac{1}{p^{(k-1)(\alpha + (k-1)\beta)}} \right)
\]
This completes the proof of Theorem 2.

References