Super Mean Labeling of Some Classes of Graphs

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Abstract: Let $G$ be a $(p, q)$ graph and $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = (f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $f^*(e) = (f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd. Then $f$ is called a super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super mean labeling is called a super mean graph. In this paper we prove that $S(P_n \circ K_1)$, $S(P_2 \times P_4)$, $S(B_{n,n})$, $\langle B_{n,n} : P_m \rangle$, $C_n \circ K_2$, $n \geq 3$, generalized antiprism $A_n^m$ and the double triangular snake $D(T_n)$ are super mean graphs.

Key Words: Smarandachely super $m$-mean labeling, Smarandachely super $m$-mean graph, super mean labeling, super mean graph.

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§1. Introduction

By a graph we mean a finite, simple and undirected one. The vertex set and the edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. The disjoint union of two graphs $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The disjoint union of $m$ copies of the graph $G$ is denoted by $mG$. The corona $G_1 \circ G_2$ of the graphs $G_1$ and $G_2$ is obtained by taking one copy of $G_1$ (with $p$ vertices) and $p$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$. Armed crown $C_n \Theta P_m$ is a graph obtained from a cycle $C_n$ by identifying the pendant vertex of a path $P_m$ at each vertex of the cycle. Bi-armed crown is a graph obtained from a cycle $C_n$ by identifying the pendant vertices of two vertex disjoint paths of equal length $m - 1$ at each vertex of the cycle. We denote a bi-armed crown by $C_n \Theta 2P_m$, where $P_m$ is a path of length $m - 1$. The double triangular snake $D(T_n)$ is the graph obtained from the path $v_1, v_2, v_3, \ldots, v_n$ by joining $v_i$ and $v_{i+1}$ with two new vertices $i_i$ and $w_i$ for $1 \leq i \leq n - 1$. The bistar $B_{m,n}$ is a graph obtained from

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The concept of super mean labeling was introduced in [7] and further discussed in [2-6]. Such a labeling is called a super mean labeling of \( G \) if \( f \) is an injection. For each edge \( e = uv \) and an integer \( m \geq 2 \), the induced Smarandachely edge \( m \)-labeling \( f^*_m \) is defined by

\[
f^*_m(e) = \left\lfloor \frac{f(u) + f(v)}{m} \right\rfloor.
\]

Then \( f \) is called a Smarandachely super \( m \)-mean labeling if \( f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, |V| + |E(G)|\} \). A graph that admits a Smarandachely super mean \( m \)-labeling is called Smarandachely super \( m \)-mean graph. Particularly, if \( m = 2 \), we know that

\[
f^*(e) = \begin{cases} 
\frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even}; \\
\frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd}.
\end{cases}
\]

Such a labeling \( f \) is called a super mean labeling of \( G \) if \( f(V(G)) \cup (f^*(e) : e \in E(G)) = \{1, 2, 3, \ldots, p + q\} \). A graph that admits a super mean labeling is called a super mean graph. The concept of super mean labeling was introduced in [7] and further discussed in [2-6].

We use the following results in the subsequent theorems.

**Theorem 2.1([7])** The bistar \( B_{m,n} \) is a super mean graph for \( m = n \) or \( n + 1 \).

**Theorem 2.2([2])** The graph \( B_{n,n} : w \), obtained by the subdivision of the central edge of \( B_{n,n} \) with a vertex \( w \), is a super mean graph.

**Theorem 2.3([2])** The bi-armed crown \( C_n \Theta 2P_m \) is a super mean graph for odd \( n \geq 3 \) and \( m \geq 2 \).

**Theorem 2.4([7])** Let \( G_1 = (p_1, q_1) \) and \( G_2 = (p_2, q_2) \) be two super mean graphs with super mean labeling \( f \) and \( g \) respectively. Let \( f(u) = p_1 + q_1 \) and \( g(v) = 1 \). Then the graph \( (G_1)_f \ast (G_2)_g \) obtained from \( G_1 \) and \( G_2 \) by identifying the vertices \( u \) and \( v \) is also a super mean graph.

## §3. Super Mean Graphs

If \( G \) is a graph, then \( S(G) \) is a graph obtained by subdividing each edge of \( G \) by a vertex.

**Theorem 3.1** The graph \( S(P_n \circ K_1) \) is a super mean graph.
Proof Let \( V(P_n \circ K_1) = \{u_i, v_i : 1 \leq i \leq n\} \). Let \( x_i(1 \leq i \leq n) \) be the vertex which divides the edge \( u_i v_i(1 \leq i \leq n) \) and \( y_i(1 \leq i \leq n - 1) \) be the vertex which divides the edge \( u_i u_{i+1}(1 \leq i \leq n - 1) \). Then \( V(S(P_n \circ K_1)) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n, 1 \leq j \leq n - 1\} \).

Define \( f : V(S(P_n \circ K_1)) \to \{1, 2, 3, \ldots, p + q = 8n - 3\} \) by

\[
\begin{align*}
f(v_1) &= 1; f(v_2) = 14; f(v_{2+i}) = 14 + 8i \text{ for } 1 \leq i \leq n - 4; \\
f(v_{n-1}) &= 8n - 11; f(v_n) = 8n - 10; f(x_1) = 3; \\
f(x_{1+i}) &= 3 + 8i \text{ for } 1 \leq i \leq n - 2; f(x_n) = 8n - 7; \\
f(u_1) &= 5; f(u_2) = 9; f(u_{2+i}) = 9i + 8i \text{ for } 1 \leq i \leq n - 3; \\
f(u_n) &= 8n - 5; f(y_1) = 8i - 1 \text{ for } 1 \leq i \leq n - 2; f(y_{n-1}) = 8n - 3.
\end{align*}
\]

It can be verified that \( f \) is a super mean labeling of \( S(P_n \circ K_1) \). Hence \( S(P_n \circ K_1) \) is a super mean graph. \( \Box \)

Example 3.2 The super mean labeling of \( S(P_5 \circ K_1) \) is given in Fig.1.

![Fig.1](image.png)

Theorem 3.2 The graph \( S(P_2 \times P_n) \) is a super mean graph.

Proof Let \( V(P_2 \times P_n) = \{u_i, v_i : 1 \leq i \leq n\} \). Let \( u_i^1, v_i^1(1 \leq i \leq n - 1) \) be the vertices which divide the edges \( u_i u_{i+1}, v_i v_{i+1}(1 \leq i \leq n - 1) \) respectively. Let \( w_i(1 \leq i \leq n) \) be the vertex which divides the edge \( u_i v_i \). That is \( V(S(P_2 \times P_n)) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{u_i^1, v_i^1 : 1 \leq i \leq n - 1\} \).

Define \( f : V(S(P_2 \times P_n)) \to \{1, 2, 3, \ldots, p + q = 11n - 6\} \) by

\[
\begin{align*}
f(u_1) &= 1; f(u_2) = 9; f(u_3) = 27; \\
f(u_i) &= f(u_{i-1}) + 5 \text{ for } 4 \leq i \leq n \text{ and } i \text{ is even} \\
f(u_i) &= f(v_{i-1}) + 17 \text{ for } 4 \leq i \leq n \text{ and } i \text{ is odd} \\
f(v_1) &= 7; f(v_2) = 16; \\
f(v_i) &= f(v_{i-1}) + 5 \text{ for } 3 \leq i \leq n \text{ and } i \text{ is odd} \\
f(v_i) &= f(v_{i-1}) + 17 \text{ for } 3 \leq i \leq n \text{ and } i \text{ is even} \\
f(w_1) &= 3; f(w_2) = 12; \\
f(w_{2+i}) &= 12 + 11i \text{ for } 1 \leq i \leq n - 2; \\
f(u_1^1) &= 6; f(u_2^1) = 24; \\
\end{align*}
\]
\[ f(u_i^1) = f(u_{i-1}^1) + 6 \text{ for } 3 \leq i \leq n - 1 \text{ and } i \text{ is odd} \]
\[ f(u_i^1) = f(u_{i-1}^1) + 16 \text{ for } 3 \leq i \leq n - 1 \text{ and } i \text{ is even} \]
\[ f(v_i^1) = 13; f(v_i^1) = f(v_{i-1}^1) + 6 \text{ for } 2 \leq i \leq n - 1 \text{ and } i \text{ is even} \]
\[ f(v_i^1) = f(v_{i-1}^1) + 16 \text{ for } 2 \leq i \leq n - 1 \text{ and } i \text{ is odd} \]

It is easy to check that \( f \) is a super mean labeling of \( S(P_2 \times P_n) \). Hence \( S(P_2 \times P_n) \) is a super mean graph. \( \square \)

**Example 3.4** The super mean labeling of \( S(P_2 \times P_0) \) is given in Fig.2.

![Fig.2](image_url)

**Theorem 3.5** The graph \( S(B_{n,n}) \) is a super mean graph.

**Proof** Let \( V(B_{n,n}) = \{u, u_i, v, v_i : 1 \leq i \leq n\} \) and \( E(B_{n,n}) = \{uu_i, vv_i, uv : 1 \leq i \leq n\} \). Let \( w, x_i, y_i, (1 \leq i \leq n) \) be the vertices which divide the edges \( uu_i, vv_i, (1 \leq i \leq n) \) respectively. Then \( V(S(B_{n,n})) = \{u, u_i, v, v_i, x_i, y_i, w : 1 \leq i \leq n\} \) and \( E(S(B_{n,n})) = \{ux_i, xu_i, uw, uv, vy_i, yv_i : 1 \leq i \leq n\} \).

Define \( f : V(S(B_{n,n})) \rightarrow \{1, 2, 3, \ldots, p + q = 8n + 5\} \) by

\[ f(u) = 1; f(x_i) = 8i - 5 \text{ for } 1 \leq i \leq n; f(u_i) = 8i - 3 \text{ for } 1 \leq i \leq n; f(w) = 8n + 3; f(v) = 8n + 5; f(y_i) = 8i - 1 \text{ for } 1 \leq i \leq n; f(v_i) = 8i + 1 \text{ for } 1 \leq i \leq n. \] It can be verified that \( f \) is a super mean labeling of \( S(B_{n,n}) \). Hence \( S(B_{n,n}) \) is a super mean graph. \( \square \)

**Example 3.6** The super mean labeling of \( S(B_{n,n}) \) is given in Fig.3.

![Fig.3](image_url)
Next we prove that the graph \( \langle B_{n,n} : P_m \rangle \) is a super mean graph. \( \langle B_{m,n} : P_k \rangle \) is a graph obtained by joining the central vertices of the stars \( K_{1,m} \) and \( K_{1,n} \) by a path \( P_k \) of length \( k - 1 \).

**Theorem 3.7** The graph \( \langle B_{n,n} : P_m \rangle \) is a super mean graph for all \( n \geq 1 \) and \( m > 1 \).

**Proof** Let \( V(\langle B_{n,n} : P_m \rangle) = \{ u_i,v_i,u,v,w_j : 1 \leq i \leq n, 1 \leq j \leq m \} \) with \( u = w_1, v = w_m \) and \( E(\langle B_{n,n} : P_m \rangle) = \{ uu_i,vv_i,w_jw_{j+1} : 1 \leq i \leq n, 1 \leq j \leq m - 1 \} \).

**Case 1** \( n \) is even.

**Subcase 1** \( m \) is odd.

By Theorem 2.2, \( \langle B_{n,n} : P_3 \rangle \) is a super mean graph. For \( m > 3 \), define \( f : V(\langle B_{n,n} : P_m \rangle) \rightarrow \{1,2,3,\ldots,p+q=4n+2m-1\} \) by

\[
\begin{align*}
   f(u) &= 1; f(u_i) = 4i - 1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n}{2} + 1; \\
   f(u_{\frac{n}{2}+1}) &= 2n + 2; f(v_i) = 4i + 1 \text{ for } 1 \leq i \leq n; f(v) = 4n + 3; \\
   f(w_2) &= 4n + 4; f(w_3) = 4n + 9; \\
   f(w_3+i) &= 4n + 9 + 4i \text{ for } 1 \leq i \leq \frac{m-5}{2}; f(w_{\frac{m+3}{2}+i}) = 4n + 2m - 4 \\
   f(w_{\frac{m+3}{2}+i}) &= 4n + 2m - 4 - 4i \text{ for } 1 \leq i \leq \frac{m-5}{2}.
\end{align*}
\]

It can be verified that \( f \) is a super mean labeling of \( \langle B_{n,n} : P_m \rangle \).

**Subcase 2** \( m \) is even.

By Theorem 2.1, \( \langle B_{n,n} : P_2 \rangle \) is a super mean graph. For \( m > 2 \), define \( f : V(\langle B_{n,n} : P_m \rangle) \rightarrow \{1,2,3,\ldots,p+q=4n+2m-1\} \) by

\[
\begin{align*}
   f(u) &= 1; f(u_i) = 4i - 1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n}{2} + 1; \\
   f(u_{\frac{n}{2}+1}) &= 2n + 2; f(v_i) = 4i + 1 \text{ for } 1 \leq i \leq n; f(v) = 4n + 3; \\
   f(w_2) &= 4n + 4; f(w_{2+i}) = 4n + 4 + 2i \text{ for } 1 \leq i \leq \frac{m-4}{2}; \\
   f(w_{\frac{m+1}{2}}) &= 4n + m + 3; \\
   f(w_{\frac{m+1}{2}+i}) &= 4n + m + 3 + 2i \text{ for } 1 \leq i \leq \frac{m-4}{2}.
\end{align*}
\]

It can be verified that \( f \) is a super mean labeling of \( \langle B_{n,n} : P_m \rangle \).

**Case 2** \( n \) is odd.

**Subcase 1** \( m \) is odd.

By Theorem 2.1, \( \langle B_{n,n} : P_2 \rangle \) is a super mean graph. For \( m > 2 \), define \( f : V(\langle B_{n,n} : P_m \rangle) \rightarrow \)
\( \{1, 2, 3, \ldots, p + q = 4n + 2m - 1\} \) by
\[
\begin{align*}
 f(u) &= 1; f(v) = 4n + 3; f(u_i) = 4i - 1 \text{ for } 1 \leq i \leq n; \\
 f(v_i) &= 4i + 1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n + 1}{2}; \\
 f(v_{\frac{m}{2} + i}) &= 2n + 2; f(w_2) = 4n + 4; \\
 f(w_{2+i}) &= 4n + 4 + 2i \text{ for } 1 \leq i \leq \frac{m-4}{2}; f(w_{\frac{m+3}{2}}) = 4n + m + 3; \\
 f(w_{\frac{m+3}{2}+i}) &= 4n + m + 3 + 2i \text{ for } 1 \leq i \leq \frac{m-4}{2}.
\end{align*}
\]

It can be verified that \( f \) is a super mean labeling of \( \langle B_{n,n} : P_m \rangle \).

Subcase 2 \( m \) is even.

By Theorem 2.2, \( \langle B_{n,n} : P_3 \rangle \) is a super mean graph. For \( m > 3 \), define \( f : V((B_{n,n} : P_m)) \to \{1, 2, 3, \ldots, p + q = 4n + 2m - 1\} \) by
\[
\begin{align*}
 f(u) &= 1; f(v) = 4n + 3; f(u_i) = 4i - 1 \text{ for } 1 \leq i \leq n; \\
 f(v_i) &= 4i + 1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n + 1}{2}; f(v_{\frac{m}{2} + i}) = 2n + 2; \\
 f(w_2) &= 4n + 4; f(w_{3}) = 4n + 9; \\
 f(w_{3+i}) &= 4n + 9 + 4i \text{ for } 1 \leq i \leq \frac{m-5}{2}; f(w_{\frac{m+3}{2}}) = 4n + 2m - 4; \\
 f(w_{\frac{m+3}{2}+i}) &= 4n + 2m - 4 - 2i \text{ for } 1 \leq i \leq \frac{m-5}{2}.
\end{align*}
\]

It can be verified that \( f \) is a super mean labeling of \( \langle B_{n,n} : P_m \rangle \). Hence \( \langle B_{n,n} : P_m \rangle \) is a super mean graph for all \( n \geq 1 \) and \( m > 1 \).

Example 3.8 The super mean labeling of \( \langle B_{4,4} : P_3 \rangle \) is given in Fig.4.

![Fig.4](image)

**Theorem 3.9** The corona graph \( C_n \odot K_2 \) is a super mean graph for all \( n \geq 3 \).

Proof Let \( V(C_n) = \{u_1, u_2, \ldots, u_n\} \) and \( V(C_n \odot K_2) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \). Then \( E(C_n \odot K_2) = \{u_iu_{i+1}, u_iu_1, u_jv_j, u_jw_j : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n\} \).

Case 1 \( n \) is odd.
The proof follows from Theorem 2.3 by taking \( m = 2 \).

**Case 2** \( n \) is even.

Take \( n = 2k \) for some \( k \). Define \( f : V(C_n \odot \overline{K}_2) \rightarrow \{1, 2, 3, \ldots, p + q = 6n\} \) by

\[
\begin{align*}
  f(u_i) &= 6i - 3 \text{ for } 1 \leq i \leq k - 1; \quad f(u_k) = 6k - 2; \\
  f(u_{2k}) &= 12k - 9; \quad f(v_i) = 6i - 5 \text{ for } 1 \leq i \leq k - 1; \quad f(v_k) = 6k - 6; \\
  f(v_{k+1}) &= 6k + 2; \quad f(v_{k+1+i}) = 6k + 2 + 6i \text{ for } 1 \leq i \leq k - 3; \quad f(v_{2k-1}) = 12k; \\
  f(v_{2k}) &= 12k - 6; \quad f(w_i) = 6i - 1 \text{ for } 1 \leq i \leq k - 1; \quad f(w_k) = 6k; \\
  f(w_{k+i}) &= 6k + 6i \text{ for } 1 \leq i \leq k - 2; \quad f(w_{2k-1}) = 12k - 4; \\
  f(w_{2k}) &= 12k - 11. 
\end{align*}
\]

It can be verified that \( f(V) \cup (f^*(e) : e \in E) = \{1, 2, 3, \ldots, 6n\} \). Hence \( C_n \odot \overline{K}_2 \) is a super mean graph.

**Example 3.10** The super mean labeling of \( C_8 \odot \overline{K}_2 \) is given in Fig.5.

![Fig.5](image)

**Theorem 3.11** The double triangular snake \( D(T_n) \) is a super mean graph.

*Proof* We prove this result by induction on \( n \). A super mean labeling of \( G_1 = D(T_2) \) is given in Fig.6.

![Fig.6](image)
Therefore the result is true for $n = 2$. Let $f$ be the super mean labeling of $G_1$ as in the above figure. Now $D(T_3) = (G_1)_f \ast (G_1)_f$, by Theorem 2.4, $D(T_3)$ is a super mean graph. Therefore the result is true for $n = 3$. Assume that $D(T_{n-1})$ is a super mean graph with the super mean labeling $g$. Now by Theorem 2.4, $(D(T_{n-1}))_g \ast (G_1)_f = D(T_n)$ is a super mean graph. Therefore the result is true for $n$. Hence by induction principle the result is true for all $n$. Thus $D(T_n)$ is a super mean graph.

\begin{proof}

Let $V(A^m_n) = \{v^j_1 : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(A^m_n) = \{v^j_1v^j_{i+1}, v^j_nv^j_1 : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v^j_1v^j_{i+1}, v^j_{i+1}v^j_{i+2}v^j_{i+1} : 2 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{v^j_1v^j_{i+1} : 1 \leq i \leq n$ and $1 \leq j \leq m-1\}.

**Case 1** $n$ is odd. Define $f : V(A^m_n) \rightarrow \{1, 2, 3, \ldots, p + q = 4mn - 2n\}$ by

\[
f(v^i_1) = 4(j-1)n + 2i - 1 \text{ for } 1 \leq i \leq \frac{n+1}{2} \text{ and } 1 \leq j \leq m;
f(v^i_{n+1}) = 4(j-1)n + n + 3 \text{ for } 1 \leq j \leq m;
f(v^i_{n+2} + 1) = 4(j-1)n + n + 3 + 2i \text{ for } 1 \leq i \leq \frac{n-3}{2} \text{ and } 1 \leq j \leq m.
\]

Then $f$ is a super mean labeling of $A^m_n$. Hence $A^m_n$ is a super mean graph.

**Case 2** $n$ is even and $n \neq 4$.

Define $f : V(A^m_n) \rightarrow \{1, 2, 3, \ldots, p + q = 4mn - 2n\}$ by

\[
f(v^i_1) = 4(j-1)n + 1 \text{ for } 1 \leq j \leq m; f(v^i_3) = 4(j-1)n + 3 \text{ for } 1 \leq j \leq m;
f(v^i_3) = 4(j-1)n + 7 \text{ for } 1 \leq j \leq m; f(v^i_4) = 4(j-1)n + 12 \text{ for } 1 \leq j \leq m;
f(v^i_{4i+1}) = 4(j-1)n + 12 + 4i \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n-6}{2};
f(v^i_{4i+2} + 1) = 4(j-1)n + 2n + 1 - 4i \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n-6}{2};
f(v^i_{n-1}) = 4(j-1)n + 9 \text{ for } 1 \leq j \leq m; f(v^i_n) = 4(j-1)n + 6 \text{ for } 1 \leq j \leq m.
\]
Then \( f \) is a super mean labeling of \( A_n^m \). Hence \( A_n^m \) is a super mean graph. \( \square \)

**Example 3.14** The super mean labeling of \( A_6^3 \) is given in Fig. 8.

**References**


