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On the Smarandache prime part

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Abstract For any positive integer n, the Smarandache Superior Prime Part $P_p(n)$ is the smallest prime number greater than or equal to n; For any positive integer $n \geq 2$, the Smarandache Inferior Prime Part $p_p(n)$ is the largest prime number less than or equal to n. The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of $\frac{S_n}{I_n}$, and give an interesting asymptotic formula for it, where $I_n = \{p_p(2) + p_p(3) + \cdots + p_p(n)\}/n$ and $S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n$.

Keywords Smarandache superior prime part, Smarandache inferior prime part, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer $n \geq 1$, the Smarandache Superior Prime Part $P_p(n)$ is defined as the smallest prime number greater than or equal to n. For example, the first few values of $P_p(n)$ are $P_p(1) = 2$, $P_p(2) = 2$, $P_p(3) = 3$, $P_p(4) = 5$, $P_p(5) = 5$, $P_p(6) = 7$, $P_p(7) = 7$, $P_P(8) = 11$, $P_p(9) = 11$, $P_p(10) = 11$, $P_p(11) = 11$, $P_p(12) = 13$, $P_p(13) = 13$, $P_p(14) = 17$, $P_p(15) = 17$, $\cdots$. For any positive integer $n \geq 2$, we also define the Smarandache Inferior Prime Part $p_p(n)$ as the largest prime number less than or equal to n. Its first few values are $p_p(2) = 2$, $p_p(3) = 3$, $p_p(4) = 3$, $p_p(5) = 5$, $p_p(6) = 5$, $p_p(7) = 7$, $p_p(8) = 7$, $p_p(9) = 7$, $p_p(10) = 7$, $p_p(11) = 11$, $\cdots$. In the book "Only problems, Not solutions" (see reference [1], problems 39), Professor F.Smarandache asked us to study the properties of the sequences $\{P_p(n)\}$ and $\{p_p(n)\}$. About these problems, it seems that none had studied them, at least we have not seen related results before. But these problems are very interesting and important, because there are close relationship between the Smarandache prime part and the prime distribution problem. Now we define

$$I_n = \{p_p(2) + p_p(3) + \cdots + p_p(n)\}/n,$$

and

$$S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n.$$

In problem 10 of reference [2], Kenichiro Kashihara asked us to determine:

(A). If $\lim_{n\to\infty} (S_n - I_n)$ converges or diverges. If it converges, find the limit.

(B). If $\lim_{n\to\infty} \frac{S_n}{I_n}$ converges or diverges. If it converges, find the limit.

For the problem (A), we can not make any progress at present. But for the problem (B), we have solved it completely. In fact we shall obtain a sharper result.

In this paper, we use the elementary and analytic methods to study the asymptotic properties of $\frac{S_n}{I_n}$, and give a shaper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 1$, we have the asymptotic formula

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{3}}\right).$$

From this theorem we may immediately deduce the following:

Corollary. The limit S_n/I_n converges as $n \longrightarrow \infty$, and

$$\lim_{n\to\infty} \frac{S_n}{I_n} = 1.$$

This solved the problem B of reference [2].

§2. Some lemmas

In order to complete the proof of the theorem, we need the following several lemmas.

First we have

Lemma 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{p_{n+1}\le x} (p_{n+1} - p_n)^2 \ll x^{\frac{23}{18}+\varepsilon},$$

where p_n denotes the n-th prime, ε denotes any fixed positive number.

Proof. This is a famous result due to D.R.Heath Brown [3] and [4].

Lemma 2. Let x be a positive real number large enough, then there must exist a prime P between x and $x + x^{\frac{2}{3}}$.

Proof. For any real number x large enough, let P_n denotes the largest prime with $P_n \le x$.

Then from Lemma 1, we may immediately deduce that

$$(P_n - P_{n-1})^2 \ll x^{\frac{23}{18}+\varepsilon},$$

or

$$P_n - P_{n-1} \ll x^{\frac{2}{3}}.$$

So there must exist a prime P between x and $x + x^{\frac{2}{3}}$.

This proves Lemma 2.

Lemma 3. For any real number $x > 1$, we have the asymptotic formulas

$$\sum_{n\le x} P_p(n) = \frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right),$$

and

$$\sum_{n\le x} p_p(n) = \frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right).$$

Proof. We only prove first asymptotic formula, similarly we can deduce the second one.

Let P_k denotes the k-th prime. Then from the definition of $P_p(n)$ we know that for any fixed prime P_r, there exist $P_{r+1} - P_r$ positive integer n such that $P_p(n) = P_r$.

So we have

$$\begin{aligned}\sum_{n\le x} P_p(n) &= \sum_{P_{n+1}\le x} P_n \cdot (P_{n+1} - P_n) \\ &= \frac{1}{2}\sum_{P_{n+1}\le x}\left(P_{n+1}^2 - P_n^2\right) - \frac{1}{2}\sum_{P_{n+1}\le x}(P_{n+1} - P_n)^2 \\ &= \frac{1}{2}P^2(x) - 2 - \frac{1}{2}\sum_{P_{n+1}\le x}(P_{n+1} - P_n)^2, \qquad (1)\end{aligned}$$

where $P(x)$ denotes the largest prime such that $P(x) \le x$.

From Lemma 2, we know that

$$P(x) = x + O\left(x^{\frac{2}{3}}\right). \qquad (2)$$

Now from (1), (2) and Lemma 1, we may immediately deduce that

$$\sum_{n\le x} P_p(n) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{5}{3}}\right) + O\left(x^{\frac{23}{18}+\varepsilon}\right) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{5}{3}}\right).$$

This proves the first asymptotic formula of Lemma 3.

The second asymptotic formula follows from Lemma 1, Lemma 2 and the identity

$$\begin{aligned}\sum_{n\le x} p_p(n) &= \sum_{P_n\le x} P_n \cdot (P_n - P_{n-1}) \\ &= \frac{1}{2}\sum_{P_n\le x}\left(P_n^2 - P_{n-1}^2\right) + \frac{1}{2}\sum_{p_n\le x}(P_n - P_{n-1})^2 \\ &= \frac{1}{2}P^2(x) + \frac{1}{2}\sum_{P_n\le x}(P_n - P_{n-1})^2.\end{aligned}$$

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. In fact for any positive integer $n > 1$, from Lemma 3 and the definition of I_n and S_n we have

$$I_n = \{p_p(2) + p_p(3) + \cdots + p_p(n)\}/n = \frac{1}{n}\left[\frac{1}{2}n^2 + O\left(n^{\frac{5}{3}}\right)\right] = \frac{1}{2}n + O\left(n^{\frac{2}{3}}\right), \qquad (3)$$

and

$$S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n = \frac{1}{n}\left[\frac{1}{2}n^2 + O\left(n^{\frac{5}{3}}\right)\right] = \frac{1}{2}n + O\left(n^{\frac{2}{3}}\right). \qquad (4)$$

Combining (3) and (4), we have

$$\frac{S_n}{I_n} = \frac{\frac{1}{2}n + O\left(n^{\frac{2}{3}}\right)}{\frac{1}{2}n + O\left(n^{\frac{2}{3}}\right)} = 1 + O\left(n^{\frac{-1}{3}}\right).$$

This completes the proof of Theorem.

The corollary follows from our Theorem as $n \longrightarrow \infty$.

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