Surface Embeddability of Graphs via Reductions

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Abstract: On the basis of reductions, polyhedral forms of Jordan axiom on closed curve in the plane are extended to establish characterizations for the surface embeddability of a graph.

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§1. Introduction

A drawing of a graph $G$ on a surface $S$ is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A Smarandache $\lambda^S$-drawing of $G$ on $S$ is a drawing of $G$ on $S$ with minimal intersections $\lambda^S$. Particularly, a Smarandache 0-drawing of $G$ on $S$, if existing, is called an embedding of $G$ on $S$.

The classical version of Jordan curve theorem in topology states that a single closed curve $C$ separates the sphere into two connected components of which $C$ is their common boundary. In this section, we investigate the polyhedral statements and proofs of the Jordan curve theorem.

Let $\Sigma = \Sigma(G; F)$ be a polyhedron whose underlying graph $G = (V, E)$ with $F$ as the set of faces. If any circuit $C$ of $G$ not a face boundary of $\Sigma$ has the property that there exist two proper subgraphs $In$ and $Ou$ of $G$ such that

$$In \bigcup Ou = G, \quad In \bigcap Ou = C,$$

then $\Sigma$ is said to have the first Jordan curve property, or simply write as 1-JCP. For a graph $G$, if there is a polyhedron $\Sigma = \Sigma(G; F)$ which has the 1-JCP, then $G$ is said to have the 1-JCP as well.

Of course, in order to make sense for the problems discussed in this section, we always suppose that all the members of $F$ in the polyhedron $\Sigma = \Sigma(G; F)$ are circuits of $G$.

**Theorem A** (First Jordan curve theorem) \hspace{1cm} $G$ has the 1-JCP If, and only if, $G$ is planar.

**Proof** Because of $\mathcal{H}_1(\Sigma) = 0$, $\Sigma = \Sigma(G; F)$, from Theorem 4.2.5 in [1], we know that

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Im \( \partial_2 = \text{Ker} \ \partial_1 = \mathcal{C} \), the cycle space of \( G \) and hence \( \text{Im} \ \partial_2 \supseteq F \) which contains a basis of \( \mathcal{C} \). Thus, for any circuit \( C \notin F \), there exists a subset \( D \) of \( F \) such that

\[
C = \sum_{f \in D} \partial_2 f; \quad C = \sum_{f \in F \setminus D} \partial_2 f. \tag{B}
\]

Moreover, if we write

\[
Ou = G[\bigcup_{f \in D} f]; \quad In = G[\bigcup_{f \in F \setminus D} f],
\]

then \( Ou \) and \( In \) satisfy the relations in (A) since any edge of \( G \) appears exactly twice in the members of \( F \). This is the sufficiency.

Conversely, if \( G \) is not planar, then \( G \) only have embedding on surfaces of genus not 0. Because of the existence of non contractible circuit, such a circuit does not satisfy the 1-JCP and hence \( G \) is without 1-JCP. This is the necessity. \( \square \)

Let \( \Sigma^* = \Sigma(G^*; F^*) \) be a dual polyhedron of \( \Sigma = \Sigma(G; F) \). For a circuit \( C \) in \( G \), let \( C^* = \{ e^* | \forall e \in C \} \), or say the corresponding vector in \( G_1^* \), of \( C \in G_1 \).

**Lemma 1** Let \( C \) be a circuit in \( \Sigma \). Then, \( G^* \setminus C^* \) has at most two connected components.

**Proof** Suppose \( H^* \) be a connected component of \( G^* \setminus C^* \) but not the only one. Let \( D \) be the subset of \( F \) corresponding to \( V(H^*) \). Then,

\[
C' = \sum_{f \in D} \partial_2 f \subseteq C.
\]

However, if \( \emptyset \neq C' \subseteq C \), then \( C \) itself is not a circuit. This is a contradiction to the condition of the lemma. From that any edge appears twice in the members of \( F \), there is only one possibility that

\[
C = \sum_{f \in F \setminus D} \partial_2 f.
\]

Hence, \( F \setminus D \) determines the other connected component of \( G^* \setminus C^* \) when \( C' = C \). \( \square \)

Any circuit \( C \) in \( G \) which is the underlying graph of a polyhedron \( \Sigma = \Sigma(G; F) \) is said to have the second Jordan curve property, or simply write 2-JCP for \( \Sigma \) with its dual \( \Sigma^* = \Sigma(G^*; F^*) \) if \( G^* \setminus C^* \) has exactly two connected components. A graph \( G \) is said to have the 2-JCP if all the circuits in \( G \) have the property.

**Theorem B** (Second Jordan curve theorem) A graph \( G \) has the 2-JCP if, and only if, \( G \) is planar.

**Proof** To prove the necessity. Because for any circuit \( C \) in \( G \), \( G^* \setminus C^* \) has exactly two connected components, any \( C^* \) which corresponds to a circuit \( C \) in \( G \) is a cocircuit. Since any edge in \( G^* \) appears exactly twice in the elements of \( V^* \), which are all cocircuits, from Lemma 1, \( V^* \) contains a basis of \( \text{Ker} \ \delta_1^* \). Moreover, \( V^* \) is a subset of \( \text{Im} \ \delta_0^* \). Hence, \( \text{Ker} \ \delta_1 \subseteq \text{Im} \ \delta_0 \). From Lemma 4.3.2 in [1], \( \text{Im} \ \delta_0^* \subseteq \text{Ker} \ \delta_1^* \). Then, we have \( \text{Ker} \ \delta_1^* = \text{Im} \ \delta_0^* \), i.e., \( \bar{H}_1(\Sigma^*) = 0 \). From the dual case of Theorem 4.3.2 in [1], \( G^* \) is planar and hence so is \( G \). Conversely, to
prove the sufficiency. From the planar duality, for any circuit \( C \) in \( G \), \( C^* \) is a cocircuit in \( G^* \). Then, \( G^* \setminus C^* \) has two connected components and hence \( C \) has the 2-JCP. \( \square \)

For a graph \( G \), of course connected without loop, associated with a polyhedron \( \Sigma = \Sigma(G; F) \), let \( C \) be a circuit and \( E_C \), the set of edges incident to, but not on \( C \). We may define an equivalence on \( E_C \), denoted by \( \sim_C \) as the transitive closure of that \( \forall a, b \in E_C \),

\[
a \sim_C b \iff \exists f \in F, \ (a^\alpha C(a, b)b^\beta \subset f) \\
\vee (b^\beta C(b, a)a^\alpha \subset f),
\]

where \( C(a, b) \), or \( C(b, a) \) is the common path from \( a \) to \( b \), or from \( b \) to \( a \) in \( C \cap f \) respectively. It can be seen that \( |E_C/ \sim_C | \leq 2 \) and the equality holds for any \( C \) not in \( F \) only if \( \Sigma \) is orientable.

In this case, the two equivalent classes are denoted by \( E_L = E_L(C) \) and \( E_R = E_R(C) \). Further, let \( V_L \) and \( V_R \) be the subsets of vertices by which a path between the two ends of two edges in \( E_L \) and \( E_R \) without common vertex with \( C \) passes respectively.

From the connectedness of \( G \), it is clear that \( V_L \cup V_R = V \setminus V(C) \). If \( V_L \cap V_R = \emptyset \), then \( C \) is said to have the third Jordan curve property, or simply write 3-JCP. In particular, if \( C \) has the 3-JCP, then every path from \( V_L \) to \( V_R \) (or vice versa) crosses \( C \) and hence \( C \) has the 1-JCP. If every circuit which is not the boundary of a face \( f \) of \( \Sigma(G) \), one of the underlain polyhedra of \( G \) has the 3-JCP, then \( G \) is said to have the 3-JCP as well.

**Lemma 2** Let \( C \) be a circuit of \( G \) which is associated with an orientable polyhedron \( \Sigma = \Sigma(G; F) \). If \( C \) has the 2-JCP, then \( C \) has the 3-JCP. Conversely, if \( V_L(C) \neq \emptyset \), \( V_R(C) \neq \emptyset \) and \( C \) has the 3-JCP, then \( C \) has the 2-JCP.

**Proof** For a vertex \( v^* \in V^* = V(G^*) \), let \( f(v^*) \in F \) be the corresponding face of \( \Sigma \). Suppose \( In^* \) and \( Ou^* \) are the two connected components of \( G^* \setminus C^* \) by the 2-JCP of \( C \). Then,

\[
In = \bigcup_{v^* \in In^*} f(v^*) \quad \text{and} \quad Ou = \bigcup_{v^* \in Ou^*} f(v^*)
\]

are subgraphs of \( G \) such that \( In \cup Ou = G \) and \( In \cap Ou = C \). Also, \( E_L \subset In \) and \( E_R \subset Ou \) (or vice versa). The only thing remained is to show \( V_L \cap V_R = \emptyset \). By contradiction, if \( V_L \cap V_R \neq \emptyset \), then \( In \) and \( Ou \) have a vertex which is not on \( C \) in common and hence have an edge incident with the vertex, which is not on \( C \), in common. This is a contradiction to \( In \cap Ou = C \).

Conversely, from Lemma 1, we may assume that \( G^* \setminus C^* \) is connected by contradiction. Then there exists a path \( P^* \) from \( v^*_1 \) to \( v^*_2 \) in \( G^* \setminus C^* \) such that \( V(f(v^*_1)) \cap V_L \neq \emptyset \) and \( V(f(v^*_2)) \cap V_R \neq \emptyset \). Consider

\[
H = \bigcup_{v^* \in P^*} f(v^*) \subseteq G.
\]

Suppose \( P = v_1 v_2 \cdots v_l \) is the shortest path in \( H \) from \( V_L \) to \( V_R \).

To show that \( P \) does not cross \( C \). By contradiction, assume that \( v_{i+1} \) is the first vertex of \( P \) crosses \( C \). From the shortness, \( v_i \) is not in \( V_R \). Suppose that subpath \( v_{i+1}, \cdots, v_{j-1}, \ i+2 \leq j < l \), lies on \( C \) and that \( v_j \) does not lie on \( C \). By the definition of \( E_L \), \( (v_{j-1}, v_j) \in E_L \) and
hence $v_j \in V_L$. This is a contradiction to the shortestness. However, from that $P$ does not cross $C$, $V_L \cap V_R \neq \emptyset$. This is a contradiction to the 3-JCP. \hfill \square

**Theorem C** (Third Jordan curve theorem) \hspace{1em} Let $G = (V, E)$ be with an orientable polyhedron $\Sigma = \Sigma(G; F)$. Then, $G$ has the 3-JCP if, and only if, $G$ is planar.

**Proof** \hspace{1em} From Theorem B and Lemma 2, the sufficiency is obvious. Conversely, assume that $G$ is not planar. By Lemma 4.2.6 in [1], $\text{Im} \partial_2 \subseteq \text{Ker} \partial_1 = \mathcal{C}$, the cycle space of $G$. By Theorem 4.2.5 in [1], $\text{Im} \partial_2 \subset \text{Ker} \partial_1$. Then, from Theorem B, there exists a circuit $C \in \mathcal{C} \setminus \text{Im} \partial_2$ without the 2-JCP. Moreover, we also have that $V_L \neq \emptyset$ and $V_R \neq \emptyset$. If otherwise $V_L = \emptyset$, let $D = \{ f | \exists e \in E_L, e \in f \} \subset F$. Because $V_L = \emptyset$, any $f \in D$ contains only edges and chords of $C$, we have

$$C = \sum_{f \in D} \partial_2 f$$

that contradicts to $C \notin \text{Im} \partial_2$. Therefore, from Lemma 2, $C$ does not have the 3-JCP. The necessity holds. \hfill \square

§2 \hspace{1em} **Reducibilities**

For $S_g$ as a surface (orientable, or nonorientable) of genus $g$, If a graph $H$ is not embedded on a surface $S_g$ but what obtained by deleting an edge from $H$ is embeddable on $S_g$, then $H$ is said to be reducible for $S_g$. In a graph $G$, the subgraphs of $G$ homeomorphic to $H$ are called a type of reducible configuration of $G$, or shortly a reduction. Robertson and Seymour in [2] has been shown that graphs have their types of reductions for a surface of genus given finite. However, even for projective plane the simplest nonorientable surface, the types of reductions are more than 100 [3,7].

For a surface $S_g, g \geq 1$, let $\mathcal{H}_{g-1}$ be the set of all reductions of surface $S_{g-1}$. For $H \in \mathcal{H}_{g-1}$, assume the embeddings of $H$ on $S_g$ have $\phi$ faces. If a graph $G$ has a decomposition of $\phi$ subgraphs $H_i, 1 \leq i \leq \phi$, such that

$$\bigcup_{i=1}^{\phi} H_i = G; \bigcup_{i \neq j} (H_i \cap H_j) = H; \quad (1)$$

all $H_i, 1 \leq i \leq \phi$, are planar and the common vertices of each $H_i$ with $H$ in the boundary of a face, then $G$ is said to be with the reducibility 1 for the surface $S_g$.

Let $\Sigma^* = (G^*; F^*)$ be a polyhedron which is the dual of the embedding $\Sigma = (G; F)$ of $G$ on surface $S_g$. For surface $S_{g-1}$, a reduction $H \subseteq G$ is given. Denote $H^* = [e^* | e \in E(H)]$. Naturally, $G^* - E(H^*)$ has at least $\phi = |F|$ connected components. If exact $\phi$ components and each component planar with all boundary vertices are successively on the boundary of a face, then $\Sigma$ is said to be with the reducibility 2.

A graph $G$ which has an embedding with reducibility 2 then $G$ is said to be with reducibility 2 as well.
Given $\Sigma = (G; F)$ as a polyhedron with under graph $G = (V, E)$ and face set $F$. Let $H$ be a reduction of surface $S_{p-1}$ and, $H \subseteq G$. Denote by $C$ the set of edges on the boundary of $H$ in $G$ and $E_C$, the set of all edges of $G$ incident to but not in $H$. Let us extend the relation $\sim_C$:

$$\forall a, b \in E_C, \quad a \sim_C b \iff \exists f \in F_H, \quad a, b \in \partial_2 f$$

(2)

by transitive law as an equivalence. Naturally, $|E_C/\sim_C| \leq \phi_H$. Denote by $\{E_i|1 \leq i \leq \phi_C\}$ the set of equivalent classes on $E_C$. Notice that $E_i = \emptyset$ can be missed without loss of generality.

Let $V_i, 1 \leq i \leq \phi_C$, be the set of vertices on a path between two edges of $E_i$ in $G$ avoiding boundary vertices. When $E_i = \emptyset, V_i = \emptyset$ is missed as well. By the connectedness of $G$, it is seen that

$$\bigcup_{i=1}^{\phi_C} V_i = V - V_H.$$ 

(3)

If for any $1 \leq i < j \leq \phi_C, V_i \cap V_j = \emptyset$, and all $[V_i]$ planar with all vertices incident to $E_i$ on the boundary of a face, then $H, G$ as well, is said to be with reducibility 3.

§3. Reducibility Theorems

**Theorem 1** A graph $G$ can be embedded on a surface $S_g(g \geq 1)$ if, and only if, $G$ is with the reducibility 1.

**Proof** Necessity. Let $\mu(G)$ be an embedding of $G$ on surface $S_g(g \geq 1)$. If $H \in \mathcal{H}_{g-1}$, then $\mu(H)$ is an embedding on $S_g(g \geq 1)$ as well. Assume $\{f_i|1 \leq i \leq \phi\}$ is the face set of $\mu(H)$, then $G_i = [\partial f_i + E([f_i]_{in})], 1 \leq i \leq \phi$, provide a decomposition satisfied by (1). Easy to show that all $G_i, 1 \leq i \leq \phi$, are planar. And, all the common edges of $G_i$ and $H$ are successively in a face boundary. Thus, $G$ is with reducibility 1.

Sufficiency. Because of $G$ with reducibility 1, let $H \in \mathcal{H}_{g-1}$, assume the embedding $\mu(H)$ of $H$ on surface $S_g$ has $\phi$ faces. Let $G$ have $\phi$ subgraphs $H_i, 1 \leq i \leq \phi$, satisfied by (1), and all $H_i$ planar with all common edges of $H_i$ and $H$ in a face boundary. Denote by $\mu_i(H_i)$ a planar embedding of $H_i$ with one face whose boundary is in a face boundary of $\mu(H)$, $1 \leq i \leq \phi$. Put each $\mu_i(H_i)$ in the corresponding face of $\mu(H)$, an embedding of $G$ on surface $S_g(g \geq 1)$ is then obtained.

**Theorem 2** A graph $G$ can be embedded on a surface $S_g(g \geq 1)$ if, and only if, $G$ is with the reducibility 2.

**Proof** Necessity. Let $\mu(G) = \Sigma = (G; F)$ be an embedding of $G$ on surface $S_g(g \geq 1)$ and $\mu^*(G) = \mu(G^*) = (G^*, F^*) (= \Sigma^*)$, its dual. Given $H \subseteq G$ as a reduction. From the duality between the two polyhedra $\mu(H)$ and $\mu^*(H)$, the interior domain of a face in $\mu(H)$ has at least a vertex of $G^*, G^* - E(H^*)$ has exactly $\phi = |F_{\mu(H)}|$ connected components. Because of each component on a planar disc with all boundary vertices successively on the boundary of the disc, $H$ is with the reducibility 2. Hence, $G$ has the reducibility 2.

Sufficiency. By employing the embedding $\mu(H)$ of reduction $H$ of $G$ on surface $S_g(g \geq 1)$ with reducibility 2, put the planar embedding of the dual of each component of $G^* - E(H^*)$ in
the corresponding face of $\mu(H)$ in agreement with common boundary, an embedding of $\mu(G)$ on surface $S_g(g \geq 1)$ is soon done. 

\[ \square \]

**Theorem 3** A 3-connected graph $G$ can be embedded on a surface $S_g(g \geq 1)$ if, and only if, $G$ is with reducibility 3.

**Proof** Necessity. Assume $\mu(G) = (G, F)$ is an embedding of $G$ on surface $S_g(g \geq 1)$. Given $H \subseteq G$ as a reduction of surface $S_{g-1}$. Because of $H \subseteq G$, the restriction $\mu(H)$ of $\mu(G)$ on $H$ is also an embedding of $H$ on surface $S_g(g \geq 1)$. From the 3-connectedness of $G$, edges incident to a face of $\mu(H)$ are as an equivalent class in $E_G$. Moreover, the subgraph determined by a class is planar with boundary in coincidence, i.e., $H$ has the reducibility 3. Hence, $G$ has the reducibility 3.

Sufficiency. By employing the embedding $\mu(H)$ of the reduction $H$ in $G$ on surface $S_g(g \geq 1)$ with the reducibility 3, put each planar embedding of $[V_i]$ in the interior domain of the corresponding face of $\mu(H)$ in agreement with the boundary condition, an embedding $\mu(G)$ of $G$ on $S_g(g \geq 1)$ is extended from $\mu(H)$.

\[ \square \]

§4. Research Notes

A. On the basis of Theorems 1–3, the surface embeddability of a graph on a surface (orientable or nonorientable) of genus smaller can be easily found with better efficiency.

For an example, the sphere $S_0$ has its reductions in two class described as $K_{3,3}$ and $K_5$. Based on these, the characterizations for the embeddability of a graph on the torus and the projective plane has been established in [4]. Because of the number of distinct embeddings of $K_5$ and $K_{3,3}$ on torus and projective plane much smaller as shown in the Appendix of [5], the characterizations can be realized by computers with an algorithm much efficiency compared with the existences, e.g., in [7].

B. The three polyhedral forms of Jordan closed planar curve axiom as shown in section 2 initiated from Chapter 4 of [6] are firstly used for surface embeddings of a graph in [4]. However, characterizations in that paper are with a mistake of missing the boundary conditions as shown in this paper.

C. The condition of 3-connectedness in Theorem 3 is not essential. It is only for the simplicity in description.

D. In all of Theorem 1–3, the conditions on planarity can be replaced by the corresponding Jordan curve property as shown in section 2 as in [4] with the attention of the boundary conditions.

References


