Total Dominator Colorings in Paths

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Abstract: Let \( G \) be a graph without isolated vertices. A total dominator coloring of a graph \( G \) is a proper coloring of the graph \( G \) with the extra property that every vertex in the graph \( G \) properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of \( G \) is called the total dominator chromatic number of \( G \) and is denoted by \( \chi_{td}(G) \). In this paper we determine the total dominator chromatic number in paths. Unless otherwise specified, \( n \) denotes an integer greater than or equal to 2.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely \( k \)-domination coloring, Smarandachely \( k \)-dominator chromatic number.

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§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [2].

Let \( G = (V, E) \) be a graph of order \( n \) with minimum degree at least one. The open neighborhood \( N(v) \) of a vertex \( v \in V(G) \) consists of the set of all vertices adjacent to \( v \). The closed neighborhood of \( v \) is \( N[v] = N(v) \cup \{v\} \). For a set \( S \subseteq V \), the open neighborhood \( N(S) \) is defined to be \( \cup_{v \in S} N(v) \), and the closed neighborhood of \( S \) is \( N[S] = N(S) \cup S \). A subset \( S \) of \( V \) is called a dominating (total dominating) set if every vertex in \( V - S \) ( \( V \) ) is adjacent to some vertex in \( S \). A dominating (total dominating) set is minimal dominating (total dominating) set if no proper subset of \( S \) is a dominating (total dominating) set of \( G \). The domination number \( \gamma \) (total domination number \( \gamma_t \)) is the minimum cardinality taken over all minimal dominating (total dominating) sets of \( G \). A \( \gamma \)-set (\( \gamma_t \)-set) is any minimal dominating (total dominating) set with cardinality \( \gamma \) (\( \gamma_t \)).

A proper coloring of \( G \) is an assignment of colors to the vertices of \( G \), such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of \( G \) is called chromatic number of \( G \) and is denoted by \( \chi(G) \). Let \( V = \{u_1, u_2, u_3, \ldots, u_p\} \) and \( \mathcal{C} = \{C_1, C_2, C_3, \ldots, C_n\} \) be a collection of subsets \( C_i \subset V \). A color represented in a vertex \( u \) is called a non-repeated color if there exists one color class \( C_i \in \mathcal{C} \) such that \( C_i = \{u\} \).

Let \( G \) be a graph without isolated vertices. A total dominator coloring of a graph \( G \) is

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a proper coloring of the graph $G$ with the extra property that every vertex in the graph $G$ properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of $G$ is called the \textit{total dominator chromatic number} of $G$ and is denoted by $\chi_{td}(G)$. Generally, for an integer $k \geq 1$, a \textit{Smarandachely $k$-dominator coloring} of $G$ is a proper coloring on $G$ such that every vertex in the graph $G$ properly dominates a $k$ color classes and the smallest number of colors for which there exists a Smarandachely $k$-dominator coloring of $G$ is called the \textit{Smarandachely $k$-dominator chromatic number} of $G$, denoted by $\chi_{S td}(G)$. Clearly, if $k = 1$, such a Smarandachely 1-dominator coloring and Smarandachely 1-dominator chromatic number are nothing but the total dominator coloring and total dominator chromatic number of $G$.

In this paper we determine total dominator chromatic number in paths.

Throughout this paper, we use the following notations.

\textbf{Notation 1.1} Usually, the vertices of $P_n$ are denoted by $u_1, u_2, \ldots, u_n$ in order. We also denote a vertex $u_i \in V(P_n)$ with $i > \lceil \frac{n}{2} \rceil$ by $u_{i-(n+1)}$. For example, $u_{n-1}$ by $u_2$. This helps us to visualize the position of the vertex more clearly.

\textbf{Notation 1.2} For $i < j$, we use the notation $\langle [i, j] \rangle$ for the subpath induced by $\langle u_i, u_{i+1}, \ldots, u_j \rangle$. For a given coloring $C$ of $P_n$, $C([i, j])$ refers to the coloring $C$ restricted to $\langle [i, j] \rangle$.

We have the following theorem from [1].

\textbf{Theorem 1.3} \textit{For any graph $G$ with $\delta(G) \geq 1$, $\max \{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G)$.}

\textbf{Definition 1.4} \textit{We know from Theorem 1.3 that $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n) + 1, \gamma_t(P_n) + 2\}$. We call the integer $n$, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1, \chi_{td}(P_n) = \gamma_t(P_n)$).}

\textbf{§2. Determination of $\chi_{td}(P_n)$}

First, we note the values of $\chi_{td}(P_n)$ for small $n$. Some of these values are computed in Theorems 2.7, 2.8 and the remaining can be computed similarly.

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<thead>
<tr>
<th>$n$</th>
<th>$\gamma_t(P_n)$</th>
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Thus $n = 2, 3, 6$ are very bad integers and we shall show that these are the only bad integers.
First, we prove a result which shows that for large values of $n$, the behavior of $\chi_{td}(P_n)$ depends
only on the residue class of $n \mod 4$ [More precisely, if $n$ is good, $m > n$ and $m \equiv n(\mod 4)$
then $m$ is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their
respective residue classes. This therefore classifies the good integers.

**Fact 2.1** Let $1 < i < n$ and let $C$ be a td-coloring of $P_n$. Then, if either $u_i$ has a repeated
color or $u_{i+2}$ has a non-repeated color, $C[\langle i + 1, n \rangle]$ is also a td-coloring. This fact is used
extensively in this paper.

**Lemma 2.2** $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$.

*Proof* For $2 \leq n \leq 5$, this is directly verified from the table. We may assume $n \geq 6$.
Let $u_1, u_2, u_3, \ldots, u_{n+4}$ be the vertices of $P_{n+4}$ in order. Let $C$ be a minimal td-coloring of
$P_{n+4}$. Clearly, $u_2$ and $u_{n+2}$ are non-repeated colors. First suppose $u_4$ is a repeated color. Then
$C[\langle 5, n + 4 \rangle]$ is a td-coloring of $P_n$. Further, $C[\langle 1, 4 \rangle]$ contains at least two color classes of $C$.
Thus $\chi_{td}(P_n + 4) \geq \chi_{td}(P_n) + 2$. Similarly the result follows if $u_{n-4}$ is a repeated color.
Thus we may assume $u_4$ and $u_{n-4}$ are non-repeated colors. But the $C[\langle 3, n + 2 \rangle]$ is a td-coloring and
since $u_2$ and $u_{n-2}$ are non-repeated colors, we have in this case also $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$. □

**Corollary 2.3** If for any $n$, $\chi_{td}(P_n) = \gamma_t(P_n) + 2$, $\chi_{td}(P_m) = \gamma_t(P_m) + 2$, for all $m > n$ with
$m \equiv n(\mod 4)$.

*Proof* By Lemma 2.2, $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2 = \gamma_t(P_n) + 2 + 2 = \gamma_t(P_{n+4}) + 2$. □

**Corollary 2.4** No integer $n \geq 7$ is a very bad integer.

*Proof* For $n = 7, 8, 9, 10$, this is verified from the table. The result then follows from the
Lemma 2.2. □

**Corollary 2.5** The integers $2, 3, 6$ are the only very bad integers.

Next, we show that $n = 8, 13, 15, 22$ are good integers. In fact, we determine $\chi_{td}(P_n)$ for
small integers and also all possible minimum td-colorings for such paths. These ideas are used
more strongly in determination of $\chi_{td}(P_n)$ for $n = 8, 13, 15, 22$.

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<tr>
<th>$n$</th>
<th>$\gamma_t(P_n)$</th>
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<td>7</td>
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Two td-colorings $C_1$ and $C_2$ of a given graph $G$ are said to be equivalent if there exists an
automorphism $f : G \rightarrow G$ such that $C_2(v) = C_1(f(v))$ for all vertices $v$ of $G$.
This is clearly an equivalence relation on the set of td-colorings of $G$. 

Total Dominator Colorings in Paths
Theorem 2.7 Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ as usual. Then

(1) $\chi_{td}(P_2) = 2$. The only minimum td-coloring is given by the color classes $\{\{u_1\}, \{u_2\}\}$

(2) $\chi_{td}(P_3) = 2$. The only minimum td-coloring is $\{\{u_1, u_3\}, \{u_2\}\}$.

(3) $\chi_{td}(P_4) = 3$ with unique minimum coloring $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}\}$.

(4) $\chi_{td}(P_5) = 4$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_5\}\}$ or $\{\{u_1, u_3\}, \{u_2\}, \{u_3\}, \{u_4\}\}$, or $\{\{u_1\}, \{u_2\}, \{u_4\}, \{u_3, u_5\}\}$.

(5) $\chi_{td}(P_6) = 4$ with unique minimum coloring $\{\{u_1, u_3\}, \{u_4, u_5\}, \{u_2\}, \{u_6\}\}$.

(6) $\chi_{td}(P_7) = 5$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$ or $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$, or $\{\{u_1\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$.

Proof We prove only (vi). The rest are easy to prove. Now, $\gamma_t(P_7) = \lceil \frac{n}{2} \rceil = 4$. Clearly $\chi_{td}(P_7) \geq 4$. We first show that $\chi_{td}(P_7) \neq 4$. Let $C$ be a td-coloring of $P_7$ with 4 colors. The vertices $u_2$ and $u_{-2} = u_6$ must have non-repeated colors. Suppose now that $u_3$ has a repeated color. Then $\{u_1, u_2, u_3\}$ must contain two color classes and $C|\{4, 7\}$ must be a td-coloring which will require at least 3 new colors (by (3)). Hence $u_3$ and similarly $u_{-3}$ must be non-repeated colors. But, then we require more than 4 colors. Thus $\chi_{td}(P_7) = 5$. Let $C$ be a minimal td-coloring of $P_7$. Let $u_2$ and $u_{-2}$ have colors 1 and 2 respectively. Suppose that both $u_3$ and $u_{-3}$ are non-repeated colors. Then, we have the coloring $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$. If either $u_3$ or $u_{-3}$ is a repeated color, then the coloring $C$ can be verified to be equivalent to the coloring given by $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$, or by $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$. □

We next show that $n = 8, 13, 15, 22$ are good integers.

Theorem 2.8 $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ if $n = 8, 13, 15, 22$.

Proof As usual, we always adopt the convention $V(P_n) = \{u_1, u_2, \ldots, u_n\}; u_{-i} = u_{n+1-i}$ for $i \geq \lceil \frac{n}{2} \rceil; C$ denotes a minimum td-coloring of $P_n$.

We have only to prove $|C| > \gamma_t(P_n) + 1$. We consider the following four cases.

Case 1 $n = 8$

Let $|C| = 5$. Then, as before $u_2$, being the only vertex dominated by $u_1$ has a non-repeated color. The same argument is true for $u_{-2}$ also. If now $u_3$ has a repeated color, $\{u_1, u_2, u_3\}$ contains 2-color classes. As $C|\{4, 8\}$ is a td-coloring, we require at least 4 more colors. Hence, $u_3$ and similarly $u_{-3}$ must have non-repeated colors. Thus, there are 4 singleton color classes and $\{u_2\}, \{u_3\}, \{u_{-2}\}$ and $\{u_{-3}\}$. The two adjacent vertices $u_4$ and $u_{-4}$ contribute two more colors. Thus $|C|$ has to be 6.

Case 2 $n = 13$

Let $|C| = 8 = \gamma_t(P_{13}) + 1$. As before $u_2$ and $u_{-2}$ are non-repeated colors. Since $\chi_{td}(P_{10}) = 7 + 2 = 9$, $u_3$ cannot be a repeated color, arguing as in case (i). Thus, $u_3$ and $u_{-3}$ are also non-repeated colors. Now, if $u_1$ and $u_{-1}$ have different colors, a diagonal of the color classes chosen
as \( \{u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3}, \ldots\} \) form a totally dominating set of cardinality \( 8 = \gamma_t(P_{13}) + 1 \).

However, clearly \( u_1 \) and \( u_{-1} \) can be omitted from this set without affecting total dominating set giving \( \gamma_t(P_{13}) \leq 6 \), a contradiction. Thus, \( u_1 \) and \( u_{-1} = u_{13} \) have the same color say 1. Thus, \( \langle [4, -4]\rangle = \langle [4, 10]\rangle \) is colored with 4 colors including the repeated color 1. Now, each of the pair of vertices \( \{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\} \) contains a color classes. Thus \( u_9 = u_{-5} \) must be colored with 1. Similarly, \( u_3 \). Now, if \( \{u_4, u_6\} \) is not a color class, the vertex with repeated color must be colored with 1 which is not possible, since an adjacent vertex \( u_5 \) which also has color 1. Therefore \( \{u_4, u_6\} \) is a color class. Similarly \( \{u_8, u_{10}\} \) is also a color class. But then, \( u_7 \) will not dominate any color class. Thus \(|C| = 9\).

Case 3 \( n = 15 \)

Let \(|C| = 9\). Arguing as before, \( u_2, u_{-2}, u_3 \) and \( u_{-3} \) have non-repeated colors \( \chi_{td}(P_{12}) = 8\): \( u_1 \) and \( u_{-1} \) have the same color, say 1. The section \( \langle [4, -4]\rangle = \langle [4, 12]\rangle \) consisting of 9 vertices is colored by 5 colors including the color 1. An argument similar to the one used in Case (2), gives \( u_4 \) and \( u_{-4} \) must have color 1. Thus, \( C| [5, -5]\rangle \) is a td-coloring with 4 colors including 1. Now, the possible minimum td-coloring of \( P_7 \) are given by Theorem 2.7. We can check that 1 can not occur in any color class in any of the minimum colorings given. e.g. take the coloring given by \( \{u_5, u_8\}, \{u_6\}, \{u_7\}, \{u_9, u_{11}\}, \{u_{10}\} \). If \( u_6 \) has color 1, \( u_5 \) can not dominate a color class. Since \( u_4 \) has color 1, \( \{u_5, u_8\} \) can not be color class 1 and so on. Thus \( \chi_{td}(P_{15}) = 10\).

Case 4 \( n = 22 \)

Let \(|C| = \gamma_t(P_{22}) + 1 = 13\). We note that \( \chi_{td}(P_{19}) = \gamma_t(P_{19}) + 2 = 12\). Then, arguing as in previous cases, we get the following facts.

Fact 1 \( u_2, u_{-2}, u_3, u_{-3} \) have non-repeated colors.

Fact 2 \( u_1 \) and \( u_{-1} \) have the same color, say 1.

Fact 3 \( u_7 \) is a non-repeated color.

This follows from the facts, otherwise \( C| [8, 22]\rangle \) will be a td-coloring: The section \( [1, 7]\rangle \) contain 4 color classes which together imply \( \chi_{td}(P_{22}) \geq 4 + \chi_{td}(P_{15}) = 4 + 10 = 14\). In particular \( \{u_5, u_7\} \) is not a color class.

Fact 4 The Facts 1 and 2, it follows that \( C| [4, -4]\rangle = C| [4, 19]\rangle \) is colored with 9 colors including 1. Since each of the pair \( \{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}, \{u_9, u_{11}\}, \{u_{12}, u_{14}\}, \{u_{13}, u_{15}\}, \{u_{16}, u_{18}\}, \{u_{17}, u_{19}\} \) contain a color class, if any of these pairs is not a color class, one of the vertices must have a non-repeated color and the other colored with 1. From Fact 3, it then follows that the vertex \( u_5 \) must be colored with 1. It follows that \( \{u_4, u_6\} \) must be a color class, since otherwise either \( u_4 \) or \( u_6 \) must be colored with 1.

Since \( \{u_4, u_6\} \) is a color class, \( u_7 \) must dominate the color class \( \{u_8\} \).

We summarize:

- \( u_2, u_3, u_7, u_8 \) have non-repeated colors.
- \( \{u_4, u_6\} \) is a color class
• $u_1$ and $u_5$ are colored with color 1.

Similarly,

• $u_{-2}, u_{-3}, u_{-7}, u_{-8}$ have non-repeated colors.

• $\{u_{-4}, u_{-6}\}$ is a color class.

• $u_{-1}$ and $u_{-5}$ are colored with color 1.

Thus the section $\langle [9, -9] \rangle = \langle [9, 14] \rangle$ must be colored with 3 colors including 1. This is easily seen to be not possible, since for instance this will imply both $u_{13}$ and $u_{14}$ must be colored with color 1. Thus, we arrive at a contradiction. Thus $\chi_{td}(P_{22}) = 14$.

**Theorem 2.9** Let $n$ be an integer. Then,

1. any integer of the form $4k$, $k \geq 2$ is good;
2. any integer of the form $4k + 1$, $k \geq 3$ is good;
3. any integer of the form $4k + 2$, $k \geq 5$ is good;
4. any integer of the form $4k + 3$, $k \geq 3$ is good.

**Proof** The integers $n = 2, 3, 6$ are very bad and $n = 4, 5, 7, 9, 10, 11, 14, 18$ are bad.

**Remark 2.10** Let $C$ be a minimal td-coloring of $G$. We call a color class in $C$, a non-dominated color class (n-d color class) if it is not dominated by any vertex of $G$. These color classes are useful because we can add vertices to these color classes without affecting td-coloring.

**Lemma 2.11** Suppose $n$ is a good number and $P_n$ has a minimal td-coloring in which there are two non-dominated color classes. Then the same is true for $n + 4$ also.

**Proof** Let $C_1, C_2, \ldots, C_r$ be the color classes for $P_n$ where $C_1$ and $C_2$ are non-dominated color classes. Suppose $u_n$ does not have color $C_1$. Then $C_1 \cup \{u_{n+1}\}, C_2 \cup \{u_{n+4}\}, \{u_{n+2}\}, \{u_{n+3}\}, C_3, C_4, \ldots, C_r$ are required color classes for $P_{n+4}$. i.e. we add a section of 4 vertices with middle vertices having non-repeated colors and end vertices having $C_1$ and $C_2$ with the coloring being proper. Further, suppose the minimum coloring for $P_n$, the end vertices have different colors. Then the same is true for the coloring of $P_{n+4}$ also. If the vertex $u_1$ of $P_n$ does not have the color $C_2$, the new coloring for $P_{n+4}$ has this property. If $u_1$ has color $C_2$, then $u_n$ does not have the color $C_2$. Therefore, we can take the first two color classes of $P_{n+4}$ as $C_1 \cup \{u_{n+4}\}$ and $C_2 \cup \{u_{n+1}\}$.

**Corollary 2.12** Let $n$ be a good number. Then $P_n$ has a minimal td-coloring in which the end vertices have different colors. [It can be verified that the conclusion of the corollary is true for all $n \neq 3, 4, 11$ and 18].

**Proof** We claim that $P_n$ has a minimum td-coloring in which: (1) there are two non-dominated color classes; (2) the end vertices have different colors.
Now, it follows from the Lemma 2.11 that (1) and (2) are true for every good integer.

Corollary 2.13 Let $n$ be a good integer. Then, there exists a minimum td-coloring for $P_n$ with two $n$-d color classes.

References