Total Dominator Colorings in Cycles

A. Vijayalekshmi
(S.T. Hindu College, Nagercoil, Tamil Nadu-629 002, India)

E-mail: vijimath.a@gmail.com

Abstract: Let $G$ be a graph without isolated vertices. A total dominator coloring of a graph $G$ is a proper coloring of $G$ with the extra property that every vertex in $G$ properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of $G$ is called the total dominator chromatic number of $G$ and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in cycles.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely $k$-dominator coloring, Smarandachely $k$-dominator chromatic number.

AMS(2010): 05C15, 05C69

§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3]. Let $G = (V,E)$ be a graph of order $n$ with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to $v$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$.

A subset $S$ of $V$ is called a total dominating set if every vertex in $V$ is adjacent to some vertex in $S$. A total dominating set is minimal total dominating set if no proper subset of $S$ is a total dominating set of $G$. The total domination number $\gamma_t$ is the minimum cardinality taken over all minimal total dominating sets of $G$. A $\gamma_t$-set is any minimal total dominating set with cardinality $\gamma_t$.

A proper coloring of $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of $G$ is called chromatic number of $G$ and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \ldots, u_p\}$ and $\mathcal{C} = \{C_1, C_2, C_3, \ldots, C_n\}$, $n \leq p$ be a collection of subsets $C_i \subset V$. A color represented in a vertex $u$ is called a non-repeated color if there exists one color class $C_i \in \mathcal{C}$ such that $C_i = \{u\}$.

Let $G$ be a graph without isolated vertices. For an integer $k \geq 1$, a Smarandachely $k$-dominator coloring of $G$ is a proper coloring of $G$ with the extra property that every vertex...
in $G$ properly dominates a $k$-color classes and the smallest number of colors for which there exists a Smarandachely $k$-dominator coloring of $G$ is called the Smarandachely $k$-dominator chromatic number of $G$ and is denoted by $\chi_{td}^k(G)$. A total dominator coloring of a graph $G$ is a proper coloring of $G$ with the extra property that every vertex in $G$ properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of $G$ is called the total dominator chromatic number of $G$ and is denoted by $\chi_{td}(G)$. In this paper, we determine total dominator chromatic number in cycles.

Throughout this paper, we use the following notations.

**Notation 1.1** Usually, the vertices of $C_n$ are denoted by $u_1, u_2, \ldots, u_n$ in order. For $i < j$, we use the notation $\langle [i, j] \rangle$ for the subpath induced by $\{u_i, u_{i+1}, \ldots, u_j\}$. For a given coloring $C$ of $C_n$, $C([i, j])$ refers to the coloring $C$ restricted to $\langle [i, j] \rangle$.

We have the following theorem from [1].

**Theorem 1.2([1])** Let $G$ be any graph with $\delta(G) \geq 1$. Then

$$\max\{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G).$$

**Definition 1.3** We know from Theorem (1.2) that $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n) + 1, \gamma_t(P_n) + 2\}$. We call the integer $n$, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1$, $\chi_{td}(P_n) = \gamma_t(P_n)$).

First, we prove a result which shows that for large values of $n$, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n \mod 4$ [More precisely, if $n$ is good, $m > n$ and $m \equiv n \mod 4$ then $m$ is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

**Fact 1.4** Let $1 < i < n$ and let $C$ be a td-coloring of $P_n$. Then, if either $u_i$ has a repeated color or $u_{i+2}$ has a non-repeated color, $C|\langle [i + 1, n] \rangle$ is also a td-coloring. This fact is used extensively in this paper.

§2. Determination of $\chi_{td}(C_n)$

It is trivially true that $\chi_{td}(C_3) = 3$ and $\chi_{td}(C_4) = 2$. We assume $n \geq 5$.

**Lemma 2.1** If $P_n$ has a minimum td-coloring in which the end vertices have different colors, then $\chi_{td}(C_n) \leq \chi_{td}(P_n)$.

*Proof* Join $u_1u_n$ by an edge and we get an induced td-coloring of $C_n$. \qed

**Corollary 2.2** $\chi_{td}(C_n) \leq \chi_{td}(P_n)$ for \forall $n \neq 3, 11, 18$.

**Lemma 2.3** If $C_n$ has a minimal td-coloring in which either there exists a color class of the form $N(x)$, where $x$ is a non-repeated color or no color class of the form $N(x)$, then
\( \chi_{td}(P_n) \leq \chi_{td}(C_n) \).

**Proof** We have assumed \( n > 3 \). If \( n = 3 \), conclusion is trivially true. We have the following two cases.

**Case 1** \( C_n \) has a minimal td-coloring \( C \) in which there is a color class of the form \( N(x) \), where \( x \) is a non-repeated color. Let \( C_n \) be the cycle \( u_1u_2\ldots u_nu_1 \). Let us assume \( x = u_2 \) has a non-repeated color \( n_1 \) and \( N(x) = \{u_1, u_3\} \) is the color class of color \( r_1 \). Then \( u_{n-1} \) has a non-repeated color since \( u_n \) has to dominate a color class which must be contained in \( N(u_n) = \{u_1, u_{n-1}\} \). Thus \( C\langle[1,n]\rangle \) is a td-coloring. Thus \( \chi_{td}(P_n) \leq \chi_{td}(C_n) \).

**Case 2** There exists \( C_n \) has a minimal td-coloring which has no color class of the form \( N(x) \). It is clear from the assumption that any vertex with a non-repeated color has an adjacent vertex with non-repeated color. We consider two sub cases.

**Subcase a** There are two adjacent vertices \( u, v \) with repeated color. Then the two vertices on either side of \( u, v \) say \( u_1 \) and \( v_1 \) must have non-repeated colors. Then the removal of the edge \( uv \) leaves a path \( P_n \) and \( C\langle[1,n]\rangle \) is a td-coloring.

**Subcase b** There are adjacent vertices \( u, v \) with \( u \) (respectively \( v \)) having repeated (respectively non-repeated) color. Then consider the vertex \( u_1 \neq v \) adjacent to \( u \). We may assume \( u_1 \) has non-repeated color (because of sub case (a)). \( v_1 \) must also have a non-repeated color since \( v \) must dominate a color class and \( u \) has a repeated color. Once again, \( C\langle(C_n - uv)\rangle \) is a td-coloring and the proof is as in sub case (a). Since either sub case (a) or sub case (b) must hold, the lemma follows. \( \Box \)

**Lemma 2.4** \( \chi_{td}(C_n) = \chi_{td}(P_n) \) for \( n = 8, 13, 15, 22 \).

**proof** We prove for \( n = 22 \). By Lemma 2.1, \( \chi_{td}(P_{22}) \geq \chi_{td}(C_{22}) \). Let \( \chi_{td}(C_{22}) < \chi_{td}(P_{22}) = 14 \). Then by Lemma 2.3, \( C_{22} \) has a minimal td-coloring in which there is a color class of the form \( N(x) \), where \( x \) is a repeated color (say \( C_1 \)). Suppose \( x = u_2 \) First, we assume that the color class of \( u_2 \) is not \( N(u_1) \) or \( N(u_3) \). Then we have \( u_4, u_5, u_{22}, u_{21} \) must be non-repeated colors.

![Fig.1](image-url)
Then $C[(6, 20)]$ is a coloring (which may not be a td-coloring for the section) with 8 colors including $C_1 \Rightarrow$ The vertices $u_7$ and $u_{19}$ have the color $C_1$. (The sets $\{u_6, u_8\}, \{u_7, u_9\}, \{u_{10}, u_{12}\}, \{u_{11}, u_{13}\}, \{u_{14}, u_{16}\}, \{u_{15}, u_{17}\}, \{u_{18}, u_{20}\}$ must contain color classes. Therefore the remaining vertex $u_{19}$ must have color $C_1$. Similarly, going the other way, we get $u_7$ must have color $C_1$). Then $\{u_6, u_8\}, \{u_{18}, u_{20}\}$ are color classes and $u_9, u_{10}, u_{16}, u_{17}$ are non-repeated colors. This leads $\{11, 15\}$ to be colored with 2 colors including $C_1$, which is not possible. Hence $\chi_{td}(C_{22}) = 14 = \chi_{td}(P_{22})$. If the color class of $u_2$ is $N(u_1)$ or $N(u_3)$, the argument is similar. Proof is similar for $n = 8, 13, 15$. \[\square\]

**Lemma 2.5** Let $n$ be a good integer. Then $\chi_{td}(P_n) \leq \chi_{td}(C_n)$

**Proof** We use induction on $n$. Let $u_1, u_2, \ldots, u_n$ be vertices of $C_n$ in order. Let $C$ be a minimal td-coloring of $C_n$. For the least good integers in their respective residue classes mod 4 is 8, 13, 15, 22, the result is proved in the previous Lemma 2.4. So we may assume that the result holds for all good integers $< n$ and that $n - 4$ is also a good integer. First suppose, there exists a color class of the form $N(x)$. Let $x = u_2$. Suppose $u_2$ has a repeated color. Then we have $u_4, u_5, u_n, u_{n-1}$ must be non-repeated color. We remove the vertices $\{u_1, u_2, u_3, u_n\}$ and add an edge $u_4u_{n-1}$ in $C_n$. Therefore, we have the coloring $C[(4, n - 1)]$ is a td-coloring with colors $\chi_{td}(C_n) - 2$. Therefore, $\chi_{td}(C_n) \geq 2 + \chi_{td}(C_{n-4}) \geq 2 + \chi_{td}(P_{n-4}) = \chi_{td}(P_n)$. 

![Fig.2](image-url)

If $x$ is a non-repeated color, then by Lemma 2.3, $\chi_{td}(P_n) \leq \chi_{td}(C_n)$. If there is no color class of the form $N(x)$, then $\chi_{td}(P_n) \leq \chi_{td}(C_n)$. \[\square\]

**Theorem 2.6** $\chi_{td}(C_n) = \chi_{td}(P_n)$, for all good integers $n$.

**Proof** The result follows from Corollary 2.2 and Lemmas 2.4 and 2.5. \[\square\]

**Remark** Thus the $\chi_{td}(C_n) = \chi_{td}(P_n)$ for $n = 8, 12, 13, 15, 16, 17$ and $\forall n \geq 19$. It can be verified that $\chi_{td}(C_n) = \chi_{td}(P_n)$ for $n = 5, 6, 7, 9, 10, 14$ and that $\chi_{td}(C_n) = \chi_{td}(P_n) + 1$ for $n = 3, 11, 18$ and that $\chi_{td}(P_4) = \chi_{td}(C_4) + 1$. 


References


