On the Smarandache totient function and the Smarandache power sequence

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Abstract For any positive integer \(n\), let \(SP(n)\) denotes the Smarandache power sequence. And for any Smarandache sequence \(a(n)\), the Smarandache totient function \(St(n)\) is defined as \(\varphi(a(n))\), where \(\varphi(n)\) is the Euler totient function. The main purpose of this paper is using the elementary and analytic method to study the convergence of the function \(S_1S_2\), where

\[
S_1 = \sum_{k=1}^{n} \left( \frac{1}{St(k)} \right)^2, \quad S_2 = \left( \sum_{k=1}^{n} \frac{1}{St(k)} \right)^2,
\]

and give an interesting limit Theorem.

Keywords Smarandache power function, Smarandache totient function, convergence.

§1. Introduction and results

For any positive integer \(n\), the Smarandache power function \(SP(n)\) is defined as the smallest positive integer \(m\) such that \(n | m^m\), where \(m\) and \(n\) have the same prime divisors. That is,

\[
SP(n) = \min \left\{ m : n | m^m, \prod_{p|m} p = \prod_{p|n} p \right\}.
\]

For example, the first few values of \(SP(n)\) are: \(SP(1) = 1, SP(2) = 2, SP(3) = 3, SP(4) = 2, SP(5) = 5, SP(6) = 6, SP(7) = 7, SP(8) = 4, SP(9) = 3, SP(10) = 10, SP(11) = 11, SP(12) = 6, SP(13) = 13, SP(14) = 14, SP(15) = 15, \cdots\) In reference [1], Professor F.Smarandache asked us to study the properties of \(SP(n)\). It is clear that \(SP(n)\) is not a multiplicative function. For example, \(SP(8) = 4, SP(3) = 3, SP(24) = 6 \neq SP(3) \times SP(8)\). But for most \(n\), we have \(SP(n) = \prod_{p|n} p\), where \(\prod_{p|n}\) denotes the product over all different prime divisors of \(n\). If \(n = p^{\alpha} \cdot k \cdot p^k + 1 \leq \alpha \leq (k + 1)p^{k+1}\), then we have \(SP(n) = p^{k+1}\), where \(0 \leq k \leq \alpha - 1\). Let \(n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}\), for all \(\alpha_i (i = 1, 2, \cdots, r)\), if \(\alpha_i \leq p_i\), then \(SP(n) = \prod_{p|n} p\).

About other properties of the function \(SP(n)\), many authors had studied it, and gave some interesting conclusions. For example, in reference [4], Zhefeng Xu had studied the mean value properties of \(SP(n)\), and obtained a sharper asymptotic formula:

\[
\sum_{n \leq x} SP(n) = \frac{1}{2} x^2 \prod_{p} \left(1 - \frac{1}{p(p+1)}\right) + O \left(x^{\frac{3}{2}} + \epsilon\right),
\]
where $\epsilon$ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

On the other hand, similar to the famous Euler totient function $\varphi(n)$, Professor F.Russo defined a new arithmetical function called the Smarandache totient function $St(n) = \varphi(a(n))$, where $a(n)$ is any Smarandache sequence. Then he asked us to study the properties of these functions. At the same time, he proposed the following:

**Conjecture.** For the Smarandache power sequence $SP(k)$, $\frac{S_1}{S_2}$ converges to zero as $n \to \infty$, where $S_1 = \sum_{k=1}^{n} \left( \frac{1}{St(k)} \right)^2$, $S_2 = \left( \sum_{k=1}^{n} \frac{1}{St(k)} \right)^2$.

In this paper, we shall use the elementary and analytic methods to study this problem, and prove that the conjecture is correct. That is, we shall prove the following:

**Theorem.** For the Smarandache power function $SP(k)$, we have $\lim_{n \to \infty} \frac{S_1}{S_2} = 0$, where

$$S_1 = \sum_{k=1}^{n} \left( \frac{1}{\varphi(SP(k))} \right)^2, \quad S_2 = \left( \sum_{k=1}^{n} \frac{1}{\varphi(SP(k))} \right)^2.$$

§2. Some lemmas

To complete the proof of the theorem, we need the following two simple Lemmas:

**Lemma 1.** For any given real number $\epsilon > 0$, there exists a positive integer $N(\epsilon)$, such that for all $n \geq N(\epsilon)$, we have $\varphi(n) \geq (1 - \epsilon) \frac{c \cdot n}{\ln \ln n}$, where $c$ is a constant.

**Proof.** See reference [5].

**Lemma 2.** For the Euler totient function $\varphi(n)$, we have the asymptotic formula

$$\sum_{k \leq n} \frac{1}{\varphi(k)} = \frac{\zeta(2) \zeta(3)}{\zeta(6)} \ln n + A + O \left( \frac{\ln n}{n} \right),$$

where $A = \gamma \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n \varphi(n)} - \sum_{n=1}^{\infty} \frac{\mu^2(n) \ln n}{n \varphi(n)}$ is a constant.

**Proof.** See reference [6].

§3. Proof of the theorem

In this section, we shall prove our Theorem.

We separate all integer $k$ in the interval $[1, n]$ into two subsets $A$ and $B$ as follows: $A$ : the set of all square-free integers. $B$ : the set of other positive integers $k$ such that $k \in [1, n] \setminus A$. So we have

$$\sum_{k \leq n} \frac{1}{(\varphi(SP(k)))^2} = \sum_{k \in A} \frac{1}{(\varphi(SP(k)))^2} + \sum_{k \in B} \frac{1}{(\varphi(SP(k)))^2}.$$

From the definition of the subset $A$, we may get

$$\sum_{k \in A} \frac{1}{(\varphi(SP(k)))^2} \leq \sum_{k \in A} \prod_{p|k} \left( 1 - \frac{1}{p} \right)^2 \leq \sum_{k=1}^{\infty} \prod_{p|k} \left( 1 - \frac{1}{p} \right)^2 \ll 1.$$
By Lemma 1, we can easily get \( \frac{k}{\varphi(k)} = O(\ln \ln k) \). Note that \( \sum_{k \leq n} \frac{\mu^2(k)}{k^2} = O(1) \). And if \( k \in B \), then we can write \( k = l \cdot m \), where \( l \) is a square-free integer and \( m \) is a square-full integer. Let \( S \) denote \( \sum_{k \in B} \left( \frac{1}{\varphi(SP(k))} \right)^2 \), then from the properties of \( SP(k) \) and \( \varphi(k) \) we have

\[
S \leq \sum_{l \leq n} \frac{1}{l^2} \prod_{p|l} p^2 \prod_{p|m} \left( 1 - \frac{1}{p} \right)^2 = \sum_{m \leq n} \frac{1}{m^2} \sum_{k \leq m} \mu^2(l) \frac{l^2}{l^2} \cdot \frac{l^2 m^2}{\varphi^2(lm)} = O \left( (\ln \ln n)^2 \sum_{m \leq n} \frac{1}{m^2} \right).
\]

Let \( U(k) = \prod_{p|k} p \), then \( \sum_{m \leq n} \frac{1}{m^2} = \sum_{k \leq n} \frac{a(k)}{U^2(k)} \), where \( m \) is a square-full integer and the arithmetical function \( a(k) \) is defined as follows:

\[
a(k) = \begin{cases} 
1, & \text{if } k \text{ is a square-full integer;} \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( \frac{a(k)}{U^2(k)} \) is a multiplicative function. According to the Euler product formula (see reference [3] and [5]), we have

\[
A(s) = \sum_{k=1}^{\infty} \frac{a(k)}{U^2(k)k^s} = \prod_{p} \left( 1 + \frac{1}{p^{2s}(p^s - 1)} \right).
\]

From the Perron formulas [5], for \( b = 1 + \frac{1}{\ln n} \), \( T \geq 1 \), we have

\[
\sum_{k \leq n} \frac{a(k)}{U^2(k)} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} A(s) \frac{n^s}{s} ds + O \left( \frac{n^b \zeta(b)}{T} \right) + O \left( n \min \left( 1, \frac{\ln n}{T} \right) \right) + \frac{a(n)}{2U^2(n)}.
\]

Taking \( T = n \), we can get the estimate

\[
O \left( \frac{n^b \zeta(b)}{T} \right) + O \left( n \min \left( 1, \frac{\ln n}{T} \right) \right) + \frac{a(n)}{2U^2(n)} = O(\ln n).
\]

Because the function \( A(s) \frac{n^s}{s} \) is analytic in \( Re \ s > 0 \), taking \( c = \frac{1}{\ln n} \), then we have

\[
\frac{1}{2\pi i} \left( \int_{b-iT}^{b+iT} A(s) \frac{n^s}{s} ds + \int_{c-iT}^{c+iT} A(s) \frac{n^s}{s} ds + \int_{b+iT}^{c+iT} A(s) \frac{n^s}{s} ds + \int_{c+iT}^{b+iT} A(s) \frac{n^s}{s} ds \right) = 0.
\]

Note that \( \int_{c-iT}^{c+iT} A(s) \frac{n^s}{s} ds = O \left( \int_{-T}^{T} \frac{dy}{\sqrt{y^2 + c^2}} \right) = O(\ln n) \) and \( \int_{c-iT}^{b-iT} A(s) \frac{n^s}{s} ds = O \left( \frac{1}{\ln n} \right) \).

Similarly, \( \int_{c+iT}^{b+iT} A(s) \frac{n^s}{s} ds = O \left( \frac{1}{\ln n} \right) \). Hence,

\[
\sum_{k \leq n} \frac{a(k)}{U^2(k)} = O(\ln n).
\]
So

\[ \sum_{k \leq n} \frac{1}{\varphi(SP(k))^2} = O(\ln n \cdot (\ln \ln n)^2). \]  \hspace{1cm} (1)

Now we come to estimate \( \sum_{k \leq n} \frac{1}{\varphi(SP(k))} \), from the definition of \( SP(n) \), we may immediately get that \( SP(n) \leq n \). Let \( n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s} \) denotes the factorization of \( n \) into prime powers, then \( SP(n) = p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s} \), where \( \beta_i \geq 1 \). Therefore, we can get that

\[
\sum_{k \leq n} \frac{1}{\varphi(SP(k))} \geq \sum_{k \leq n} \frac{1}{\varphi(k)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \ln n + O\left(\frac{\ln n}{n}\right). \hspace{1cm} (2)
\]

Combining (1) and (2), we obtain

\[
0 \leq \frac{\left(\sum_{k=1}^{n} \frac{1}{\varphi(SP(k))}\right)^2}{\left(\sum_{k=1}^{n} \frac{1}{\varphi(SP(k))}\right)^2} \leq \frac{O\left(\ln n \cdot (\ln \ln n)^2\right)}{\left(\frac{\zeta(2)\zeta(3)}{\zeta(6)} \ln n + O\left(\frac{\ln n}{n}\right)\right)^2} \rightarrow 0, \hspace{1cm} \text{as } n \to \infty.
\]

This completes the proof of our Theorem.

References