Smarandachely $t$-path step signed graphs

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Abstract A Smarandachely $k$-signed graph (Smarandachely $k$-marked graph) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called underlying graph of $S$ and $\sigma : E \rightarrow (e_1, e_2, \cdots, e_k)$ ($\mu : V \rightarrow (\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_k)$) is a function, where each $e_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a signed graph or a marked graph. E. Prisner [9] in his book Graph Dynamics defines the $t$-path step operator on the class of finite graphs. Given a graph $G$ and a positive integer $t$, the $t$-path step graph $(G)_t$ of $G$ is formed by taking a copy of the vertex set $V(G)$ of $G$, joining two vertices $u$ and $v$ in the copy by a single edge $e = uv$ whenever there exists a $u-v$ path of length $t$ in $G$. Analogously, one can define the Smarandachely $t$-path step signed graph $(S)_t = ((G)_t, \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $(G)_t$ called $t$-path step graph and sign of any edge $e = uv$ in $(S)_t$ is $\mu(u)\mu(v)$. It is shown that for any signed graph $S$, its $(S)_t$ is balanced. We then give structural characterization of Smarandachely $t$-path step signed graphs. Further, in this paper we characterize signed graphs which are switching equivalent to their Smarandachely 3-path step signed graphs.

Keywords Smarandachely $k$-signed graphs, Smarandachely $k$-marked graphs, signed graphs, marked graphs, balance, switching, Smarandachely $t$-path step signed graphs, negation.

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [4]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A Smarandachely $k$-signed graph (Smarandachely $k$-marked graph) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called underlying graph of $S$ and $\sigma : E \rightarrow (\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_k)$ ($\mu : V \rightarrow (\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_k)$) is a function, where each $\bar{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a signed graph or a marked graph. A signed graph $S = (G, \sigma)$ is balanced if every cycle in $S$ has an even number of negative edges (Harary [3]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of $S$ is positive.

A marking of $S$ is a function $\mu : V(G) \rightarrow \{+, -\}$; A signed graph $S$ together with a marking $\mu$ by $S_\mu$. Given a signed graph $S$ one can easily define a marking $\mu$ of $S$ as follows:
For any vertex \( v \in V(S) \),
\[
\mu(v) = \prod_{u \in N(v)} \sigma(uv),
\]
the marking \( \mu \) of \( S \) is called canonical marking of \( S \).

The following characterization of balanced signed graphs is well known.

**Proposition 1.1.**[^6] A signed graph \( S = (G, \sigma) \) is balanced if, and only if, there exist a marking \( \mu \) of its vertices such that each edge \( uv \) in \( S \) satisfies \( \sigma(uv) = \mu(u)\mu(v) \).

Given a marking \( \mu \) of \( S \), by switching \( S \) with respect to \( \mu \) we mean reversing the sign of every edge of \( S \) whenever the end vertices have opposite signs in \( S_\mu \)^[1]. We denote the signed graph obtained in this way is denoted by \( S_\mu(S) \) and this signed graph is called the \( \mu \)-switched signed graph or just switched signed graph. A signed graph \( S_1 \) switches to a signed graph \( S_2 \) (that is, they are switching equivalent to each other), written \( S_1 \sim S_2 \), whenever there exists a marking \( \mu \) such that \( S_\mu(S_1) \cong S_2 \).

Two signed graphs \( S_1 = (G, \sigma) \) and \( S_2 = (G', \sigma') \) are said to be weakly isomorphic (Sozýnsky \[^7\] ) or cycle isomorphic (Zaslavsky \[^8\]) if there exists an isomorphism \( \phi : G \to G' \) such that the sign of every cycle \( Z \) in \( S_1 \) equals to the sign of \( \phi(Z) \) in \( S_2 \). The following result is well known:

**Proposition 1.2.**[^8] Two signed graphs \( S_1 \) and \( S_2 \) with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

### §2. Smarandachely \( t \)-path step signed graphs

Given a graph \( G \) and a positive integer \( t \), the \( t \)-path step graph \( (G)_t \) of \( G \) is formed by taking a copy of the vertex set \( V(G) \) of \( G \), joining two vertices \( u \) and \( v \) in the copy by a single edge \( e = uv \) whenever there exists a \( u-v \) path of length \( t \) in \( G \). The notion of \( t \)-path step graphs was defined in [9], page 168.

In this paper, we shall now introduce the concept of Smarandachely \( t \)-path step signed graphs as follows: The Smarandachely \( t \)-path step signed graph \( (S)_t = ((G)_t, \sigma') \) of a signed graph \( S = (G, \sigma) \) is a signed graph whose underlying graph is \( (G)_t \) called \( t \)-path step graph and sign of any edge \( e = uv \) in \( (S)_t \) is \( \mu(u)\mu(v) \), where \( \mu \) is the canonical marking of \( S \). Further, a signed graph \( S = (G, \sigma) \) is called Smarandachely \( t \)-path step signed graph, if \( S \cong (S')_t \), for some signed graph \( S' \).

The following result indicates the limitations of the notion of Smarandachely \( t \)-path step signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be Smarandachely \( t \)-path step signed graphs.

**Proposition 2.1.** For any signed graph \( S = (G, \sigma) \), its \( (S)_t \) is balanced.

**Proof.** Since sign of any edge \( e = uv \) is \( (S)_t \) is \( \mu(u)\mu(v) \), where \( \mu \) is the canonical marking of \( S \), by Proposition 1.1, \( (S)_t \) is balanced.

**Remark.** For any two signed graphs \( S \) and \( S' \) with same underlying graph, their Smarandachely \( t \)-path step signed graphs are switching equivalent.

**Corollary 2.2.** For any signed graph \( S = (G, \sigma) \), its Smarandachely 2 (3)-path step signed graph \( (S)_2 \) (\( (S)_3 \)) is balanced.
The following result characterize signed graphs which are Smarandachely $t$-path step signed graphs.

**Proposition 2.3.** A signed graph $S = (G, \sigma)$ is a Smarandachely $t$-path step signed graph if, and only if, $S$ is balanced signed graph and its underlying graph $G$ is a $t$-path step graph.

**Proof.** Suppose that $S$ is balanced and $G$ is a $t$-path step graph. Then there exists a graph $H$ such that $(H_t) \cong G$. Since $S$ is balanced, by Proposition 1.1, there exists a marking $\mu$ of $G$ such that each edge $e = uv$ in $S$ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge $e$ in $H$, $\sigma'(e)$ is the marking of the corresponding vertex in $G$. Then clearly, $(S')_t \cong S$. Hence $S$ is a Smarandachely $t$-path step signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a Smarandachely $t$-path step signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $(S')_t \cong S$. Hence $G$ is the $t$-path step graph of $H$ and by Proposition 2.1, $S$ is balanced.

§3. **Switching invariant Smarandachely 3-path step signed graphs**

Zelinka [9] prove that the graphs in Fig. 1 are all unicyclic graphs which are fixed in the operator $(G)_3$, i.e. graphs $G$ such that $G \cong (G)_3$. The symbols $p, q$ signify that the number of vertices and edges in Fig. 1.

**Proposition 3.1.**[9] Let $G$ be a finite unicyclic graph such that $G \cong (G)_3$. Then either $G$ is a circuit of length not divisible by 3, or it is some of the graphs depicted in Fig. 1.

![Fig. 1.](image)

In view of the above result, we have the following result for signed graphs:

**Proposition 3.2.** For any signed graph $S = (G, \sigma)$, $S \sim (S)_3$ if, and only if, $G$ is a cycle of length not divisible by 3, or it is some of the graphs depicted in Fig. 1 and $S$ is balanced.

**Proof.** Suppose $S \sim (S)_3$. This implies, $G \cong (G)_3$ and hence by Proposition 3.1, we see that the $G$ must be isomorphic to either $C_m$, $4 \leq m \neq 3k$, $k$ is a positive integer or the graphs depicted in Fig. 1. Now, if $S$ is any signed graph on any of these graphs, Corollary 4 implies that $(S)_3$ is balanced and hence if $S$ is unbalanced its Smarandachely 3-path step signed graph $(S)_3$ being balanced cannot be switching equivalent to $S$ in accordance with Proposition 1.2. Therefore, $S$ must be balanced.
Conversely, suppose that $S$ is a balanced signed graph on $C_m$, $4 \leq m \neq 3k$, $k$ is a positive integer or the graphs depicted in Fig. 1. Then, since $(S)_3$ is balanced as per Corollary 2.2 and since $G \cong (G)_3$ in each of these cases, the result follows from Proposition 1.2.

**Problem.** Characterize the signed graphs for which $S \cong (S)_3$.

The notion of *negation* $\eta(S)$ of a given signed graph $S$ defined by Harary $^3$ as follows: $\eta(S)$ has the same underlying graph as that of $S$ with the sign of each edge opposite to that given to it in $S$. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in $S$ while applying the unary operator $\eta(.)$ of taking the negation of $S$.

For a signed graph $S = (G, \sigma)$, the $(S)_t$ is balanced (Proposition 2.1). We now examine, the condition under which negation of $(S)_t$ (i.e., $\eta((S)_t)$) is balanced.

**Proposition 3.3.** Let $S = (G, \sigma)$ be a signed graph. If $(G)_t$ is bipartite then $\eta((S)_t)$ is balanced.

**Proof.** Since, by Proposition 2.1, $(S)_t$ is balanced, then every cycle in $(S)_t$ contains even number of negative edges. Also, since $(G)_t$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle $C$ in $(S)_t$ are also even. This implies that the same thing is true in negation of $(S)_t$. Hence $\eta((S)_t)$ is balanced.

Proposition 3.2 provides easy solutions to three other signed graph switching equivalence relations, which are given in the following results.

**Corollary 3.4.** For any signed graph $S = (G, \sigma)$, $\eta(S) \sim (S)_3$ if, and only if, $S$ is unbalanced signed graph on $C_{2m+1}$, $m \geq 2$ or first two graphs depicted in Fig. 1.

**Corollary 3.5.** For any signed graph $S = (G, \sigma)$, $(\eta(S))_3 \sim (S)_3$.

**References**