Open Distance-Pattern Uniform Graphs

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Abstract: Given an arbitrary non-empty subset $M$ of vertices in a graph $G = (V, E)$, each vertex $u$ in $G$ is associated with the set $f_M^u = \{d(u, v) : v \in M, u \neq v\}$, called its open $M$-distance-pattern. A graph $G$ is called a Smarandachely uniform $k$-graph if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f_M^u = f_M^v$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a $k$-family of open distance-pattern uniform (odpu-) set of $G$ and the minimum cardinality of odpu-sets in $G$, if they exist, is called the Smarandachely odpu-number of $G$, denoted by $\text{od}_k^S(G)$. Usually, a Smarandachely uniform 1-graph $G$ is called an open distance-pattern uniform (odpu-) graph. In this case, its odpu-number $\text{od}_k^S(G)$ of $G$ is abbreviated to $\text{od}(G)$. In this paper we present several fundamental results on odpu-graphs and odpu-number of a graph.

Key Words: Smarandachely uniform $k$-graph, open distance-pattern, open distance-pattern uniform graphs, open distance-pattern uniform (odpu-) set, Smarandachely odpu-number, odpu-number.

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§1. Introduction

All graphs considered in this paper are finite, simple, undirected and connected. For graph theoretic terminology we refer to Harary [6].

The concept of open distance-pattern and open distance-pattern uniform graphs were suggested by B.D. Acharya. Given an arbitrary non-empty subset $M$ of vertices in a graph $G = (V, E)$, the open $M$-distance-pattern of a vertex $u$ in $G$ is defined to be the set $f_M^u = \{d(u, v) : v \in M, u \neq v\}$, where $d(x, y)$ denotes the distance between the vertices $x$ and $y$ in $G$. A graph $G$ is called a Smarandachely uniform $k$-graph if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f_M^u = f_M^v$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a $k$-family of open distance-pattern uniform (odpu-) set of $G$ and the minimum cardinality of odpu-sets in $G$, if they exist, is called the Smarandachely odpu-number of $G$, denoted by $\text{od}_k^S(G)$. Usually, a Smarandachely uniform 1-graph $G$ is called an open distance-pattern uniform (odpu-) graph. In this case, its odpu-number $\text{od}_k^S(G)$ of $G$ is abbreviated to $\text{od}(G)$. We need the following theorem.
Theorem 1.1([5]) Let $G$ be a graph of order $n, n \geq 4$. Then the following conditions are equivalent.

(i) The graph $G$ is self-centred with radius $r \geq 2$ and for every $u \in V(G)$, there exists exactly one vertex $v$ such that $d(u, v) = r$.

(ii) The graph $G$ is $r$-decreasing.

(iii) There exists a decomposition of $V(G)$ into pairs $\{u, v\}$ such that $d(u, v) = r(G) > \max(d(u, x), d(x, v))$ for every $x \in V(G) - \{u, v\}$.

In this paper we present several fundamental results on odpu-graphs and odpu-number of a graph $G$.

8. Odpu-Sets in Graphs

It is clear that an odpu-set in any nontrivial graph must have at least two vertices. The following theorem gives a basic property of odpu-sets.

Theorem 2.1 In any graph $G$, if there exists an odpu-set $M$, then $M \subseteq Z(G)$ where $Z(G)$ is the center of the graph $G$. Also $M \subseteq Z(G)$ is an odpu-set if and only if $f_M^o(v) = \{1, 2, \ldots, r(G)\}$, for all $v \in V(G)$.

Proof Let $G$ have an odpu-set $M \subseteq V(G)$ and let $v \in M$. Suppose $v \notin Z(G)$. Then $e(v) > r(G)$. Hence there exists a vertex $u \in V(G)$ such that $d(u, v) > r(G)$. Since $v \in M$, $f_M^o(u)$ contains an element, which is greater than $r(G)$. Now let $w \in V(G)$ be such that $e(w) = r(G)$. Then $d(w, v) \leq r(G)$ for all $v \in M$. Hence $f_M^o(w)$ does not contain an element greater than $r(G)$, so that $f_M^o(u) \neq f_M^o(w)$. Thus $M$ is not an odpu-set, which is a contradiction. Hence $M \subseteq Z(G)$.

Now, let $M \subseteq Z(G)$ be an odpu-set. Then $\max f_M^o(v) = r(G)$. Let $u \in M$ be such that $d(u, v) = r(G)$. Let the shortest $u - v$ path be $(u = v_1, v_2, \ldots, v_{r(G)} = v)$. Then $v_1$ is adjacent to $u$. Therefore, $1 \in f_M^o(v_1)$. Since $M$ is an odpu-set, $1 \in f_M^o(x)$ for all $x \in V(G)$. Now, $d(v_2, u) = 2$, whence $2 \in f_M^o(v_2)$. Since $M$ is an odpu-set, $2 \in f_M^o(x)$ for all $x \in V(G)$. Proceeding like this, we get $\{1, 2, 3, \ldots, r(G)\} \subseteq f_M^o(x)$ and since $M \subseteq Z(G)$, $f_M^o(x) = \{1, 2, 3, \ldots, r(G)\}$ for all $x \in V$. The converse is obvious.

Corollary 2.2 A connected graph $G$ is an odpu-graph if and only if the center $Z(G)$ of $G$ is an odpu-set.

Proof Let $G$ be an odpu-graph with an odpu-set $M$. Then $f_M^o(v) = \{1, 2, \ldots, r(G)\}$ for all $v \in V(G)$. Since $f_M^o(v) \supseteq f_M^o(v)$ and $d(u, v) \leq r(G)$ for every $v \in V$ and $u \in Z(G)$, it follows that $Z(G)$ is an odpu set of $G$. The converse is obvious.

Corollary 2.3 Every self-centered graph is an odpu-graph.
**Proof** Let \( G \) be a self-centered graph. Take \( M = V(G) \). Since \( G \) is self-centered, \( e(v) = r(G) \) for all \( v \in V(G) \). Therefore, \( f_M^o(v) = \{1, 2, \cdots, r(G)\} \) for all \( v \in V(G) \), so that \( M \) is an odpu-set for \( G \). \( \Box \)

**Remark** 2.4 The converse of Corollary 2.3 is not true. For example the graph \( K_2 + \overline{K}_2 \), is not self-centered but it is an odpu-graph. Moreover, there exist self-centered graphs having a proper subset of \( Z(G) = V(G) \) as an odpu-set.

**Theorem** 2.5 If \( G \) is an odpu-graph with \( n \geq 3 \), then \( \delta(G) \geq 2 \) and \( G \) is 2-connected.

**Proof** Let \( G \) be an odpu-graph with \( n \geq 3 \) and let \( M \) be an odpu-set of \( G \). If \( G \) has a pendant vertex \( v \), it follows from Theorem 2.1 that \( v \notin M \). Also, \( v \) is adjacent to exactly one vertex \( w \in V(G) \). Since \( M \) is an odpu-set, \( \max f_M^o(w) = r(G) \). Therefore, there exists \( u \in M \) such that \( d(u, w) = r(G) \). Now \( d(u, v) = r(G) + 1 \) and \( f_M^o(v) \) contains \( r(G) + 1 \). Hence \( f_M^o(v) \neq f_M^o(w) \), a contradiction. Thus \( \delta(G) \geq 2 \).

Now suppose \( G \) is not 2-connected. Let \( B_1 \) and \( B_2 \) be blocks in \( G \) such that \( V(B_1) \cap V(B_2) = \{u\} \). Since, the center of a graph lies in a block, we may assume that the center \( Z(G) \subseteq B_1 \). Let \( v \in B_2 \) be such that \( wv \in E(G) \). Then there exists a vertex \( w \in M \) such that \( d(u, w) = r(G) \) and \( d(v, w) = r(G) + 1 \), so that \( r(G) + 1 \in f_M^o(u) \), which is a contradiction. Hence \( G \) is 2-connected. \( \Box \)

**Corollary** 2.6 A tree \( T \) has an odpu-set \( M \) if and only if \( T \) is isomorphic to \( P_2 \).

**Corollary** 2.7 If \( G \) is a unicyclic odpu-graph, then \( G \) is isomorphic to a cycle.

**Corollary** 2.8 A block graph \( G \) is an odpu-graph if and only if \( G \) is complete.

**Corollary** 2.9 In any graph \( G \), if there exists an odpu-set \( M \), then every subset \( M' \) of \( Z(G) \) such that \( M \subseteq M' \) is also an odpu-set.

Thus Corollary 2.9 shows that in a limited sense the property of subsets of \( V(G) \) being odpu-sets is super-hereditary within \( Z(G) \). The next remark gives an algorithm to recognize odpu-graphs.

**Remark** 2.10 Let \( G \) be a finite simple connected graph. The following algorithm recognizes odpu-graphs.

**Step-1:** Determine the center of the graph \( G \).

**Step-2:** Generate the \( c \times n \) distance matrix \( D(G) \) of \( G \) where \( c = |Z(G)| \).

**Step-3:** Check whether each column \( C_i \) has the elements \( 1, 2, \ldots, r \).

**Step-4:** If then, \( G \) is an odpu-graph.

Or else \( G \) is not an odpu-graph.

The above algorithm is efficient since we have polynomial time algorithm to determine \( Z(G) \) and to compute the matrix \( D(G) \).
Theorem 2.11 Every odpu-graph $G$ satisfies, $r(G) \leq d(G) \leq r(G) + 1$. Further for any positive integer $r$, there exists an odpu-graph with $r(G) = r$ and $d(G) = r + 1$.

Proof Let $G$ be an odpu-graph. Since $r(G) \leq d(G)$ for any graph $G$, it is enough to prove that $d(G) \leq r(G) + 1$. If $G$ is a self-centered graph, then $r(G) = d(G)$. Assume $G$ is not self-centered and let $u$ and $v$ be two antipodal vertices of $G$. Since $G$ is an odpu-graph, $Z(G)$ is an odpu-set and hence there exist vertices $u', v' \in Z(G)$ such that $d(u, u') = 1$ and $d(v, v') = 1$. Now, $G$ is not self-centered, and $d(u, v) = d$ implies $u, v \notin Z(G)$. If $d > r + 1$; since $d(u, u') = d(v, v') = 1$, the only possibility is $d(u', v') = r$, which implies $d(u, v') = r + 1$. But $v' \in Z(G)$ and hence $r + 1 \in f^*_M(u)$, which is not possible. Hence $d(u, v) = d \leq r + 1$ and the result follows.

Now, let $r$ be any positive integer. For $r = 1$ take $G = K_2 + \overline{K}_n$, $n \geq 2$. For $r \geq 2$, let $G$ be the graph obtained from $C_{2r}$ by adding a vertex $v_{r}$ corresponding to each edge $e$ in $C_{2r}$ and joining $v_{r}$ to the end vertices of $e$. Then, it is easy to check that an odpu-set of the resulting graph is $V(C_{2r})$.

However, it should be noted that $d = r + 1$ is not a sufficient condition for the graph to be an odpu-graph. For the graph $G$ consisting of the cycle $C_{r}$ with exactly one pendant edge at one of its vertices, $d = r + 1$ but $G$ is not an odpu-graph.

Remark 2.12 Theorem 2.11 states that there are only two classes of odpu-graphs, those which are self-centered or those for which $d(G) = r(G) + 1$. Hence, the problem of characterizing odpu-graphs reduces to the problem of characterizing odpu-graphs with $d(G) = r(G) + 1$.

The following theorem gives a complete characterization of odpu-graphs with radius one.

Theorem 2.13 A graph with radius $1$ and diameter $2$ is an odpu-graph if and only if there exists a subset $M \subset V(G)$ with $|M| \geq 2$ such that the induced subgraph $\langle M \rangle$ is complete, $\langle V - M \rangle$ is not complete and any vertex in $V - M$ is adjacent to all the vertices of $M$.

Proof Assume that $G$ is an odpu-graph with radius $r = 1$ and diameter $d = 2$. Then, $f^*_M(v) = \{1\}$ for all $v \in V(G)$. If $\langle M \rangle$ is not complete, then there exist two vertices $u, v \in M$ such that $d(u, v) \geq 2$. Hence, both $f^*_M(u)$ and $f^*_M(v)$ contains a number greater than 1, which is not possible. Therefore, $\langle M \rangle$ is complete. Next, if $x \in V - M$ then, since $f^*_M(x) = \{1\}$, $x$ is adjacent to all the vertices of $\langle M \rangle$. Now, if $\langle V - M \rangle$ is complete, then since $\langle M \rangle$ is complete the above argument implies that $G$ is complete, whence diameter of $G$ would be one, a contradiction. Thus, $\langle V - M \rangle$ is not complete.

Conversely assume $\langle M \rangle$ is complete with $|M| \geq 2$, $\langle V - M \rangle$ is not complete and every vertex of $\langle V - M \rangle$ is adjacent to all the vertices in $\langle M \rangle$. Then, clearly, the diameter of $G$ is two and radius of $G$ is one. Also, since $|M| \geq 2$, there exist at least two universal vertices in $M$ (i.e. each is adjacent to every other vertices in $M$). Therefore $f^*_M(v) = \{1\}$ for every $v \in V(G)$. Hence $G$ must be an odpu-graph with $M$ as an odpu-set.

Theorem 2.14 Let $G$ be a graph of order $n \geq 3$. Then the following are equivalent.

(i) Every $k$-element subset of $V(G)$ forms an odpu-set, where $2 \leq k \leq n$. 


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(ii) Every 2-element subset of \( V(G) \) forms an odpu-set.

(iii) \( G \) is complete.

Proof Trivially (i) implies (ii)

If every 2-element subset \( M \) of \( V(G) \) forms an odpu-set, then \( f_M^o(v) = \{1\} \) for all \( v \in V(G) \) and hence \( G \) is complete.

Obviously (iii) implies (i). \( \square \)

Theorem 2.15 Any graph \( G \) (may or may not be connected) with \( \delta(G) \geq 1 \) and having no vertex of full-degree can be embedded into an odpu-graph \( H \) with \( G \) as an induced subgraph of \( H \) of order \( |V(G)| + 2 \) such that \( V(G) \) is an odpu-set of the graph \( H \).

Proof Let \( G \) be a graph with \( \delta(G) \geq 1 \) and having no vertex of full-degree. Let \( u, v \in V(G) \) be any two adjacent vertices and let \( a, b \notin V(G) \). Let \( H \) be the graph obtained by joining \( a \) to \( b \) and also, joining \( a \) to all vertices of \( G \) except \( u \) and joining the vertex \( b \) to all vertices of \( G \) except \( v \). Let \( M = V(G) \subset V(H) \). Since \( a \) is adjacent to all the vertices except \( u \) and \( d(a, u) = 2 \), implies \( f_M^o(a) = \{1, 2\} \). Similarly \( f_M^o(b) = \{1, 2\} \). Since \( u \) is adjacent to \( v \), \( 1 \in f_M^o(u) \). Since \( u \) does not have full degree, there exists a vertex \( x \), which is not adjacent to \( u \). But \( (u, b, x) \) is a path in \( H \) and hence \( d(u, x) = 2 \) in \( H \) for all such \( x \in V(G) \). Therefore \( f_M^o(u) = \{1, 2\} \). Similarly \( f_M^o(v) = \{1, 2\} \). Now let \( w \in V(G) \), \( w \neq u, v \). Now since no vertex \( w \) is an isolated vertex and \( w \) does not have full degree, there exist vertices \( x \) and \( y \) in \( V(G) \) such that \( wx \in E(H) \) and \( wy \notin E(H) \). But then, there exists a path \( (w, a, y) \) or \( (w, b, y) \) with length 2 in \( H \). Also every vertex which is not adjacent to \( w \) is at a distance 2 in \( H \). Therefore \( f_M^o(w) = \{1, 2\} \). Hence \( f_M^o(x) = \{1, 2\} \) for all \( x \in V(H) \). Hence \( H \) is an odpu-graph and \( V(G) \) is an odpu-set of \( H \). \( \square \)

Remark 2.16 Bollobás [1] proved that almost all graphs have diameter 2 and almost no graph has a node of full degree. Hence almost no graph has radius one. Since \( r(G) \leq d(G) \), almost all graphs have \( r(G) = d(G) = 2 \), that is, almost all graphs are self-centered with diameter 2. Since self-centered graphs are odpu-graphs, the following corollary is immediate.

Corollary 2.17 Almost all graphs are odpu-graphs.

§3. Odpu-Number of a Graph

As we have observed in section 2, if \( G \) has an odpu-set \( M \) then \( M \subseteq Z(G) \) and if \( M \subseteq M' \subseteq Z(G) \), then \( M' \) is also an odpu-set. This motivates the definition of odpu-number of an odpu-graph.

Definition 3.1 The Odpu-number of a graph \( G \), denoted by \( od(G) \), is the minimum cardinality of an odpu-set in \( G \).

In this section we characterize odpu-graphs which have odpu-number 2 and also prove that
there is no graph with odpu-number 3 and for any positive integer \( k \neq 1, 3 \), there exists a graph with odpu-number \( k \). We also present several embedding theorems. Clearly,

\[
2 \leq \text{od}(G) \leq |Z(G)| \quad \text{for any odpu} - \text{graph } G. \quad (3.1)
\]

Since the upper bound for \( |Z(G)| \) is \( |V(G)| \), the above inequality becomes,

\[
2 \leq \text{od}(G) \leq |V(G)|. \quad (3.2)
\]

The next theorem gives a characterization of graphs attaining the lower bound in the above inequality.

**Theorem 3.2** For any graph \( G \), \( \text{od}(G) = 2 \) if and only if there exist at least two vertices \( x, y \in V(G) \) such that \( d(x) = d(y) = |V(G)| - 1 \).

**Proof** Suppose that the graph \( G \) has an odpu-set \( M \) with \( |M| = 2 \). Let \( M = \{x, y\} \). We claim that \( d(x) = d(y) = n - 1 \), where \( n = |V(G)| \). If not, there are two possibilities.

**Case 1.** \( d(x) = n - 1 \) and \( d(y) < n - 1 \).

Since \( d(x) = n - 1 \), \( x \) is adjacent to \( y \). Therefore, \( f_M^o(x) = \{1\} \). Also, since \( d(y) < n - 1 \), it follows that \( 2 \in f_M^o(w) \) for any vertex \( w \) not adjacent to \( v \), which is a contradiction.

**Case 2.** \( d(x) < n - 1 \) and \( d(y) < n - 1 \).

If \( xy \in E(G) \), then \( f_M^o(x) = f_M^o(y) = \{1\} \) and for any vertex \( w \) not adjacent to \( u \), \( f_M^o(w) \neq \{1\} \).

If \( xy \notin E(G) \), then \( 1 \notin f_M^o(x) \) and for any vertex \( w \) which is adjacent to \( x \), \( 1 \in f_M^o(w) \), which is a contradiction. Hence \( d(x) = d(y) = n - 1 \).

Conversely, let \( G \) be a graph with \( u, v \in V(G) \) such that \( d(u) = d(v) = n - 1 \). Let \( M = \{u, v\} \). Then \( f_M^o(x) = \{1\} \) for all \( x \in V(G) \) and hence \( M \) is an odpu-set with \( |M| = 2 \). \( \square \)

**Corollary 3.3** For any odpu-graph \( G \) if \( |M| = 2 \), then \( \langle M \rangle \) is isomorphic to \( K_2 \).

**Corollary 3.4** \( \text{od}(K_n) = 2 \) for all \( n \geq 2 \).

**Corollary 3.5** If a \( (p, q) \)-graph has an odpu-set \( M \) with odpu-number 2, then \( 2p - 3 \leq q \leq \frac{p(p - 1)}{2} \).

**Proof** By Theorem 3.2, there exist at least two vertices having degree \( p - 1 \) and hence \( q \geq 2p - 3 \). The other inequality is trivial. \( \square \)

**Theorem 3.6** There is no graph with odpu-number three.

**Proof** Suppose there exists a graph \( G \) with \( \text{od}(G) = 3 \) and let \( M = \{x, y, z\} \) be an odpu-set in \( G \). Since \( G \) is connected, \( 1 \in f_M^o(x) \cap f_M^o(y) \cap f_M^o(z) \).

We claim that \( x, y, z \) form a triangle in \( G \). Since \( 1 \in f_M^o(x) \), and \( 1 \in f_M^o(z) \), we may assume that \( xy, yz \in E(G) \). Now if \( xz \notin E(G) \), then \( d(x, z) = 2 \) and hence \( 2 \in f_M^o(x) \cap f_M^o(Z) \) and \( f_M^o(y) = \{1\} \), which is not possible. Thus \( xz \in E(G) \) and \( x, y, z \) forms a triangle in \( G \).
Now \( f_M^o(w) = \{1\} \) for any \( w \in V(G) - M \) and hence \( w \) is adjacent to all the vertices of \( M \). Thus \( G \) is complete and \( od(G) = 2 \), which is again a contradiction. Hence there is no graph \( G \) with \( od(G) = 3 \). \( \Box \)

Next we prove that the existence of graph with odpu-numbers \( k \neq 1,3 \). We need the following definition.

**Definition 3.7** The shadow graph \( S(G) \) of a graph \( G \) is obtained from \( G \) by adding for each vertex \( v \) of \( G \) a new vertex \( v' \), called the shadow vertex of \( v \), and joining \( v' \) to all the neighbors of \( v \) in \( G \).

**Theorem 3.8** For every positive integer \( k \neq 1,3 \), there exists a graph \( G \) with odpu-number \( k \).

**Proof** Clearly \( od(P_2) = 2 \) and \( od(C_4) = 4 \). Now we will prove that the shadow graph of any complete graph \( K_n, n \geq 3 \) is an odpu-graph with odpu-number \( n + 2 \).

Let the vertices of the complete graph \( K_n \) be \( v_1, v_2, \ldots, v_n \) and the corresponding shadow vertices be \( v_1', v_2', \ldots, v_n' \). Since the shadow graph \( S(K_n) \) of \( K_n \) is self-centered with radius 2 and \( n \geq 3 \), by Corollary 2.3, it is an odpu-graph. Let \( M \) be the smallest odpu-set of \( S(K_n) \).

We establish that \( |M| = n + 2 \) in the following three steps.

First, we show \( \{v_1', v_2', \ldots, v_n'\} \subseteq M \). If there is a shadow vertex \( v'_i \notin M \), then \( 2 \notin f_M^o(v_i) \) since \( v_i \) is adjacent to all the vertices of \( S(K_n) \) other than \( v_i' \), implying thereby that \( M \) is not an odpu-set, contrary to our assumption. Thus, the claim holds.

Now, we show that \( M = \{v_1', v_2', \ldots, v_n'\} \) is not an odpu-set of \( S(K_n) \). Note that \( v_1', v_2', \ldots, v_n' \) are pairwise non-adjacent and if \( M = \{v_1', v_2', \ldots, v_n'\} \), then \( 1 \notin f_M^o(v'_i) \) for all \( v'_i \in M \). But \( 1 \in f_M^o(v_i), \ 1 \leq i \leq n \), and hence \( M \) is not an odpu-set.

From the above two steps, we conclude that \( |M| > n \). Now, \( M = \{v_1', v_2', \ldots, v_n'\} \cup \{v_i\} \) where \( v_i \) is any vertex of \( K_n \) is not an odpu-set. Further, since all the shadow vertices are pairwise non-adjacent and \( v_i \) is not adjacent to \( v_i' \), \( 1 \notin f_M^o(v_i') \). Hence \( |M| > n \). Let \( v_i, v_j \in V(K_n) \) be any two vertices of \( K_n \) and let \( M = \{v_i, v_j, v_1', v_2', \ldots, v_n'\} \). We prove that \( M \) is an odpu-set and thereby establish that \( od(G) = n + 2 \). Now, \( d(v_i, v_j) = 1 \) and \( d(v_i, v'_j) = d(v_j, v'_i) = 2 \), so that \( f_M^o(v_i) = f_M^o(v_j) = \{1,2\} \). Also, for any vertex \( v_k \in V(K_n) \), \( d(v_k, v_i) = 1 \) and \( d(v_k, v'_i) = 2 \), so that \( f_M^o(v_k) = \{1,2\} \). Again, \( d(v'_i, v_j) = d(v'_j, v_i) = 1 \) and for any shadow vertex \( v'_k \in V(S(K_n)) \), \( d(v'_k, v_i) = 1 \) and since all the shadow vertices are pairwise non-adjacent, \( f_M^o(v'_k) = \{1,2\} \). Thus, \( M \) is an odpu-set and \( od(G) = n + 2 \). \( \Box \)

**Remark 3.9** We have proved that 3 cannot be the odpu number of any graph. Hence, by the above theorem, for an odpu-graph the numbers 1 and 3 are the only two numbers forbidden as odpu-numbers of any graph.

**Theorem 3.10** \( od(C_{2k+1}) = 2k \).

**Proof** Let \( C_{2k+1} = \{v_1, v_2, \ldots, v_{2k+1}, v_1\} \). Clearly \( M = \{v_1, v_2, \ldots, v_{2k}\} \) is an odpu-set of \( C_{2k+1} \). Now, let \( M \) be any odpu-set of \( C_{2k+1} \). Then, there exists a vertex \( v_i \in V(C_{2k+1}) \) such that \( v_i \notin M \). Without loss of generality, assume that \( v_i = v_{2k+1} \). Then, since \( 1 \notin f_M^o(v_{2k+1}) \), either \( v_{2k} \in M \) or \( v_1 \in M \) or both \( v_1, v_{2k} \in M \). Without loss of generality, let \( v_1 \in M \). Since
Theorem contains at least two vertices. \((\forall x, i \in V(G))\) such that \(d(x, v) = r(x, v)\) for all \(x \in V(G)\). We now proceed to characterize odpu-graphs \(G\) with \(od(G) = |V(G)|\). We need the following lemma.

**Lemma 3.12** Let \(G\) be a self-centered graph with \(r(G) \geq 2\). Then for each \(u \in V(G)\), there exist at least two vertices in every \(i\)th neighborhood \(N_i(u) = \{v \in V(G) : d(u, v) = i\}\) of \(u, i = 1, 2, \ldots, r - 1\).

**Proof** Let \(G\) be a self-centered graph and let \(u\) be any arbitrary vertex of \(G\). If possible, let for some \(i\), \(1 \leq i \leq r - 1\), \(N_i(u)\) contains exactly one vertex, say \(w\). Then, since \(e(w) = r\), there exists \(x \in V(G)\) such that \(d(x, w) = r\).

If \(x \in N_j(u)\) for some \(j > i\), then \(d(u, x) > r\), which is a contradiction. Again if \(x \in N_j(u)\) for some \(j < i\), then \(d(x, w) = r < i \leq r - 1\), which is again a contradiction. Hence \(N_i(u)\) contains at least two vertices. \(\square\)

**Theorem 3.13** Let \(G\) be a graph of order \(n, n \geq 4\). Then the following conditions are equivalent.

(i) \(od(G) = n\).

(ii) the graph \(G\) is self-centered with radius \(r \geq 2\) and for every \(u \in V(G)\), there exists exactly one vertex \(v\) such that \(d(u, v) = r\).

(iii) the graph \(G\) is \(r\)-decreasing.

(iv) there exists a decomposition of \(V(G)\) into pairs \(\{u, v\}\) such that \(d(u, v) = r(G) > \max(\{d(u, x), d(x, v)\})\) for every \(x \in V(G) - \{u, v\}\).

**Proof** Let \(G\) be a graph of order \(n, n \geq 4\). The equivalence of (ii), (iii) and (iv) follows from Theorem 1.1. We now prove that (i) and (ii) are equivalent.

(i) \(\Rightarrow\) (ii)

Let \(G\) be a graph with \(od(G) = n = |V(G)|\). Hence, \(e(u) = r\) for all \(u \in V(G)\) so that \(G\) is self-centered. Now, we show that for every \(u \in V(G)\), there exists exactly one vertex \(v \in V(G)\) such that \(d(u, v) = r\).

First, we show that for some vertex \(v_0 \in V(G)\), there exists exactly one vertex \(v_0 \in V(G)\) such that \(d(u_0, v_0) = r\). Suppose for every vertex \(x \in V(G)\), there exist at least two vertices \(x_1\) and \(x_2\) in \(V(G)\) such that \(d(x, x_1) = r\) and \(d(x, x_2) = r\). Let \(M = V(G) - \{x_1\}\). Then, since \(d(x, x_2) = r\), \(f_M^0(x) = \{1, 2, \ldots, r\}\). Further, since \(d(x, x_1) = r\), \(f_M^0(x_1) = \{1, 2, \ldots, r\}\). Also, since \(d(x, x_2) = r\), and by Lemma 3.12, \(f_M^0(x_2) = \{1, 2, \ldots, r\}\). Let \(y\) be any vertex other than
Let $1 \leq k \leq r$, and if $d(y,x) = k$, then by Lemma 3.12 and by assumption, there exists another vertex $z \in M$ such that $d(y,z) = k$. Therefore, $f^k_M(y) = \{1,2,\ldots,r\}$. Thus $M = V(G) - \{x_1\}$ is an odpu-set for $G$, which is a contradiction to the hypothesis. Thus, there exists a vertex $u_0 \in V(G)$ such that there is exactly one vertex $v_0 \in V(G)$ with $d(u_0,v_0) = r$. Next, we claim that $u_0$ is the unique vertex for $v_0$ such that $d(u_0,v_0) = r$. Suppose there is a vertex $w_0 \neq u_0$ with $d(w_0,v_0) = r$. Let $M = V(G) - \{u_0\}$. Then, $d(u_0,v_0) = r$ implies $f^0_M(u_0) = \{1,2,\ldots,r\}$ and $d(v_0,w_0) = r$ implies $f^r_M(v_0) = \{1,2,\ldots,r\}$. Also, since $d(v_0,w_0) = r$, by Lemma 3.12, it follows that $f^r_M(v_0) = \{1,2,\ldots,r\}$. Now let $x \in V(G) - \{u_0,v_0\}$. Since $d(x,u_0) < r$, we get $f^r_M(x) = \{1,2,\ldots,r\}$. Hence, $M = V(G) - \{u_0\}$ is an odpu-set for $G$, which is a contradiction. Therefore, for the vertex $v_0$, $u_0$ is the unique vertex such that $d(u_0,v_0) = r$.

Next, we claim that there is some vertex $u_1 \in V(G) - \{u_0,v_0\}$ such that there is exactly one vertex $v_1 \in V(G)$ at a distance $r$ from $u_1$. If for every vertex $u_1 \in V(G) - \{u_0,v_0\}$, there are at least two vertices $v_1$ and $w_1$ in $V(G)$ at a distance $r$ from $u_1$, then proceeding as above, we can prove that $M = V(G) - \{v_1\}$ is an odpu-set of $G$, a contradiction. Therefore, $v_1$ is the only vertex at a distance $r$ from $u_1$. Continuing the above procedure we conclude that for every vertex $u \in V(G)$ there exists exactly one vertex $v \in V(G)$ at a distance $r$ from $u$ and for the vertex $v$, $u$ is the only vertex at a distance $r$. Thus (i) implies (ii).

Now, suppose (ii) holds. Then $M$ is the unique odpu-set of $G$ and hence $od(G) = n$. □

**Corollary 3.14** If $G$ is an odpu-graph with $od(G) = |V(G)| = n$, then $G$ is self-centered and $n$ is even.

**Corollary 3.15** If $G$ is an odpu-graph with $od(G) = |V(G)| = n$ then $r(G) \geq 3$ and $u_1,u_2$ are different vertices of $G$, then $N(u_1) \neq N(u_2)$.

**Proof** If $N(u_1) = N(u_2)$, then $d(u_1,v_1) = d(u_2,v_1)$, which contradicts Theorem 3.13. □

**Corollary 3.16** The odpu-number $od(G) = |V(G)|$ for the $n$-dimensional cube and for even cycle $C_{2n}$.

**Corollary 3.17** Let $G$ be a graph with $r(G) = 2$. Then $od(G) = |V(G)|$ if and only if $G$ is isomorphic to $K_{2,2,\ldots,2}$.

**Proof** If $G = K_{2,2,\ldots,2}$, then $r(G) = 2$ and $G$ is self-centered and by Theorem 3.13, $od(G) = |V(G)| = 2n$.

Conversely, let $G$ be a graph with $r(G) = 2$. Then $G$ is self-centered and it follows from Theorem 3.13 that for each vertex, there exists exactly one vertex at a distance $2$. Hence $G \cong K_{2,2,\ldots,2}$.

**Problem 3.1** Characterize odpu-graphs for which $od(G) = |Z(G)|$.

**Theorem 3.18** If a graph $G$ has odpu-number 4, then $r(G) = 2$.

**Proof** Let $G$ be an odpu-graph with odpu-number 4. Let $M = \{u,v,x,y\}$ be an odpu-set of $G$. If $r(G) = 1$, then $f^1_M(x) = \{1\}$ for all $x \in V(G)$. Therefore, $\langle M \rangle$ is complete. Hence, any two elements of $M$ form an odpu-set of $G$ which implies $od(G) = 2$, which is a contradiction.
Hence $r(G) \geq 2$.

Since $r(G) \geq 2$, none of the vertices in $M$ is adjacent to all the other vertices in $M$ and $\langle M \rangle$ has no isolated vertex. Hence $\langle M \rangle = P_4$ or $C_4$ or $2K_2$.

If $\langle M \rangle = P_4$ or $C_4$ then the radius of $\langle M \rangle$ is 2. Hence, there exists a vertex $v$ in $M$ such that $f^*_M(v) = \{1, 2\}$ so that $r(G) = 2$.

Suppose $\langle M \rangle = 2P_2$ and let $E(\langle M \rangle) = \{xy, xy\}$. Since $|M| = 4$, $r(G) \leq 3$. If $r(G) = 3$, then $3 \in f^*_M(x)$ and $3 \in f^*_M(u)$. Hence, there exists a vertex $w \notin M$ such that $xw, uw \in E(G)$. Hence, $d(x, w) = d(u, w) = 1$. Also, $d(y, w) = d(v, w) = 2$. Therefore, $3 \notin f^*_M(w)$, which is a contradiction. Thus, $r(G) = 2$. □

A set $S$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex of $G$ is either in $S$ or is adjacent to a vertex in $S$; further, if $\langle S \rangle$ is isolate-free then $S$ is called a total dominating set of $G$ (see Haynes et al[7]). The next result establishes the relation between odpu-sets and total dominating sets in an odpu-graph.

**Theorem 3.19** For any odpu-graph $G$, every odpu-set in $G$ is a total dominating set of $G$.

**Proof** Let $M$ be an odpu-set of the graph $G$. Since $1 \in f^*_M(u)$, for all $u \in V(G)$, for any vertex $u \in V(G)$ there exists a vertex $v \in M$ such that $uv \in E(G)$. Hence, $M$ is a total dominating set of $G$. □

Recall that the total domination number $\gamma_t(G)$ of a graph $G$ is the least cardinality of a total dominating set in $G$.

**Corollary 3.20** For any odpu-graph $G$, $\gamma_t(G) \leq od(G)$.

**Problem 3.2** Characterize odpu-graphs $G$ such that $\gamma_t(G) = od(G)$.

Let $H$ be a graph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and let $G_1, G_2, \ldots, G_n$ be a set of vertex disjoint graphs. Then the graph obtained from $H$ by replacing each vertex $x_i$ of $H$ by the graph $G_i$ and joining all the vertices of $G_i$ to all the vertices of $G_j$ if and only if $x_i; x_j \in E(H)$, is denoted as $H[G_1, G_2, \ldots, G_n]$.

**Theorem 3.21** Let $H$ be a connected odpu-graph of order $n \geq 2$ and radius $r \geq 2$. Let $K = H[G_1, G_2, \ldots, G_n]$. Then $od(H) = od(K)$.

**Proof** Let $V(H) = \{x_1, x_2, \ldots, x_n\}$. Let $G_i$ be the graph replaced at the vertex $x_i$ in $H$. It follows from the definition of $K$ that if $(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ is a shortest path in $H$, then $(x_{i_1,j_1}, x_{i_2,j_2}, \ldots, x_{i_r,j_r})$ is a shortest path in $K$ where $x_{i_k,j_k}$ is an arbitrary vertex in $G_{i_k}$. Hence $M \subseteq V(H)$ is odpu-set in $H$ if and only if the set $M_1 \subseteq V(K)$, where $M_1$ has exactly one vertex from $G_i$ if and only if $x_i \in M$, is an odpu-set for $K$. Hence $od(H) = od(K)$. □

**Corollary 3.22** A graph $G$ with radius $r(G) \geq 2$ is an odpu-graph if and only if its shadow graph is an odpu-graph.

**Theorem 3.23** Given a positive integer $n \neq 1, 3$, any graph $G$ can be embedded as an induced subgraph into an odpu-graph $K$ with odpu-number $n$. 
Proof If \( n = 2 \), then \( K = C_3[G, K_1, K_1] \) is an odpu-graph with \( \text{od}(K) = \text{od}(C_3) = 2 \) and \( G \) is an induced subgraph of \( K \). Suppose \( n \geq 4 \). Then by Theorem 3.8, there exists an odpu-graph \( H \) with \( \text{od}(H) = n \). Now by Theorem 3.21, \( K = H[G, K_1, K_1, \ldots, K_1] \) is an odpu-graph with \( \text{od}(K) = \text{od}(H) = n \) and \( G \) is an induced subgraph of \( K \). \( \square \)

Remark 3.24 If \( G \) and \( K \) are as in Theorem 3.23, we have

(1) \( \omega(H) = \omega(G) + 2 \),
(2) \( \chi(H) = \chi(G) + 2 \),
(3) \( \beta_1(H) = \beta_1(G) + 1 \) and
(4) \( \beta_0(H) = \beta_0(G) \)

where \( \omega(G) \) is the clique number, \( \chi(G) \) is the chromatic number, \( \beta_1(G) \) is the matching number and \( \beta_0(G) \) is the independence number of \( G \). Since finding these parameters are NP-complete for graphs, finding these four parameters for an odpu-graph is also NP-complete.

§4. Bipartite Odpu-Graphs

In this section we characterize complete multipartite odpu-graphs and bipartite odpu-graphs with odpu-number 2 and 4. Further we prove that there are no bipartite graph with odpu-number 5.

Theorem 4.1 The complete \( n \)-partite graph \( K_{a_1, a_2, \ldots, a_n} \) is an odpu-graph if and only if either \( a_i = a_j = 1 \) for some \( i \) and \( j \) or \( a_1, a_2, a_3, \ldots, a_n \geq 2 \). Hence \( \text{od}(K_{a_1, a_2, \ldots, a_n}) = 2 \) or \( 2n \).

Proof Suppose \( G = K_{a_1, a_2, \ldots, a_n} \) is an odpu-graph. If \( a_i = 1 \) for exactly one \( i \), then \( |Z(K_{a_1, a_2, \ldots, a_n})| = 1 \). Hence \( G \) is not an odpu-graph, which is a contradiction.

Conversely assume, either \( a_i = a_j = 1 \) for some \( i \) and \( j \) or \( a_1, a_2, a_3, \ldots, a_n \geq 2 \). If \( a_i = a_j = 1 \) for some \( i \) and \( j \), then there exist two vertices of full degree and hence \( G \) is an odpu-graph with odpu-number 2. If \( a_1, a_2, a_3, \ldots, a_n \geq 2 \), then for any set \( M \) which contains exactly two vertices from each partite set, we have \( f_M^G(v) = \{1, 2\} \) for all \( v \in V(G) \) and hence \( M \) is an odpu-set with \( |M| = 2n \). Further if \( M \) is any subset of \( V(G) \) with \( |M| < 2n \), there exists a partite set \( V_i \) such that \( |M \cap V_i| \leq 1 \) and \( f_M^G(v) = \{1\} \) for some \( v \in V_i \) and \( M \) is not an odpu-set. Hence \( \text{od}(G) = 2n \). \( \square \)

Theorem 4.2 Let \( G \) be a bipartite odpu-graph. Then \( \text{od}(G) = 2 \) if and only if \( G \) is isomorphic to \( P_2 \).

Proof Let \( G \) be a bipartite odpu-graph with bipartition \((X, Y)\). Let \( \text{od}(G) = 2 \). Then, by Theorem 3.2, there exist at least two vertices of degree \( n - 1 \). Hence \( |X| = |Y| = 1 \) and \( G \) is isomorphic to \( P_2 \). The converse is obvious. \( \square \)
Theorem 4.3 A bipartite odpu-graph $G$ with bipartition $(X, Y)$ has odpu-number 4 if and only if the set $X$ has at least two vertices of degree $|Y|$ and the set $Y$ has at least two vertices of degree $|X|$.

Proof Suppose $od(G) = 4$. Let $M$ be an odpu-set of $G$ with $|M| = 4$. Then, by Theorem 3.18, $r(G) = 2$ and hence $f_M^o(x) = \{1, 2\}$ for all $x \in V(G)$.

First, we show that $|M \cap X| = |M \cap Y| = 2$. If $|M \cap X| = 3$ and $|M \cap Y| = 1$ then $2 \notin f_M^o(v)$ for the vertex $v \in M \cap Y$. Hence it follows that $|M \cap X| = |M \cap Y| = 2$. Let $M \cap X = \{u, v\}$ and $M \cap Y = \{x, y\}$. Since $f_M^o(w) = \{1, 2\}$ for all $w \in V$, it follows that every vertex in $X$ is adjacent to both $x$ and $y$ and every vertex in $Y$ is adjacent to both $u$ and $v$. Hence, $deg(u) = deg(v) = |Y|$ and $deg(x) = deg(y) = |X|$.

Conversely, suppose $u, v \in X, x, y \in Y, deg(u) = deg(v) = |Y|$ and $deg(x) = deg(y) = |X|$. Let $M = \{u, v, x, y, \}$. Clearly $f_M^o(w) = \{1, 2\}$ for all $w \in V$. Hence $M$ is an odpu-set. Also, since there exists no full degree vertex in $G$, by Theorem 3.2 the odpu-number cannot be equal to 2. Also, since 3 is not the odpu-number of any graph. Hence the odpu-number of $G$ is 4. □

Theorem 4.4 The number 5 cannot be the odpu-number of a bipartite graph.

Proof Suppose there exists a bipartite graph $G$ with bipartition $(X, Y)$ and $od(G) = 5$. Let $M = \{u, v, x, y, z\}$ be a odpu-set for $G$.

First, we shall show that $|X \cap M| \geq 2$ and $|Y \cap M| \geq 2$. Suppose, on the contrary, one of these inequalities fails to hold, say $|X \cap M| \leq 1$. If $X$ has no element in $M$, then $1 \notin f_M^o(a)$ for all $a \in M$, which is a contradiction. Therefore, $|X \cap M| = 1$. Without loss of generality, let $\{u\} = X \cap M$. Then, since $1 \in f_M^o(v) \cap f_M^o(x) \cap f_M^o(y) \cap f_M^o(z)$, all the vertices $v, x, y, z$ should be adjacent to $u$. Hence $2 \notin f_M^o(u)$, a contradiction. Thus, we see that each of $X$ and $Y$ must have at least two vertices in $M$. Without loss of generality, we may assume $u, v \in X$ and $x, y, z \in Y$.

Case 1. $r(G) = 2$.

Then $f_M^o(w) = \{1, 2\}$ for all $w \in Y$. Then proceeding as in Theorem 4.3, we get $deg(u) = deg(v) = |Y|$ and $deg(x) = deg(y) = deg(z) = |X|$. Therefore, by Theorem 4.3, $\{u, v, x, y\}$ forms an odpu-set of $G$, a contradiction to our assumption that $M$ is a minimum odpu-set of $G$. Therefore, $r = 2$ is not possible.

Case 2. $r(G) \geq 3$.

Since $M$ is an odpu-set of $G$, $f_M^o(a) = \{1, 2, \ldots, r\}$ for all $a \in V(G)$. Then, $2 \notin f_M^o(u)$, there exists a vertex $b \in Y$ such that $ub, bv \in E(G)$. But since $b \in Y$ and $ub, bv \in E(G)$, $3 \notin f_M^o(b)$, which is a contradiction. Hence the result follows. □

Conjecture 4.5 For a bipartite odpu-graph the odpu-number is always even.

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