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A Note on Path Signed Digraphs

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Abstract: A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of S* and $\sigma : \mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. In this paper, we define the path signed digraph $\overrightarrow{P_k}(S) = (\overrightarrow{P_k}(D), \sigma')$ of a given signed digraph $S = (D, \sigma)$ and offer a structural characterization of signed digraphs that are switching equivalent to their 3-path signed digraphs $\overrightarrow{P_3}(S)$. The concept of a line signed digraph is generalized to that of a path signed digraphs. Further, in this paper we discuss the structural characterization of path signed digraphs $\overrightarrow{P_k}(S)$.

Key Words: Smarandachely k-Signed digraphs, Smarandachely k-marked digraphs, signed digraphs, marked digraphs, balance, switching, path signed digraphs, line signed digraphs, negation.

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§1. Introduction

For standard terminology and notion in digraph theory, we refer the reader to the classic text-books of Bondy and Murty [2]and Harary et al. [4]; the non-standard will be given in this paper as and when required.

A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of S* and σ : $\mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. A *signed digraph* is an ordered pair $S = (D, \sigma)$, where

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$D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. A *marking* of S is a function $\mu : V(D) \to \{+, -\}$. A signed digraph S together with a marking μ is denoted by S_μ. A signed digraph $S = (D, \sigma)$ is *balanced* if every semicycle of S is positive (See [4]). Equivalently, a signed digraph is balanced if every semicycle has an even number of negative arcs. The following characterization of balanced signed digraphs is obtained in [9].

Proposition 1.1(E. Sampathkumar et al. [9]) *A signed digraph $S = (D, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$.*

In [9], the authors define switching and cycle isomorphism of a signed digraph as follows:

Let $S = (D, \sigma)$ and $S' = (D', \sigma')$, be two signed digraphs. Then S and S' are said to be isomorphic, *if there exists an isomorphism $\phi : D \to D'$ (that is a bijection $\phi : V(D) \to V(D')$ such that if $\overrightarrow{uv}$ is an arc in D then $\overrightarrow{\phi(u)\phi(v)}$ is an arc in D') such that for any arc $\overrightarrow{e} \in D$, $\sigma(\overrightarrow{e}) = \sigma'(\phi(\overrightarrow{e}))$.*

Given a marking μ of a signed digraph $S = (D, \sigma)$, *switching* S with respect to μ is the operation changing the sign of every arc $\overrightarrow{uv}$ of S' by $\mu(u)\sigma(\overrightarrow{uv})\mu(v)$. The signed digraph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called μ *switched signed digraph* or just *switched signed digraph.*

Further, a signed digraph S switches to signed digraph S' (or that they are switching equivalent to each other), written as $S \sim S'$, whenever there exists a marking of S such that $\mathcal{S}_\mu(S) \cong S'$.

Two signed digraphs $S = (D, \sigma)$ and $S' = (D', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : D \to D'$ such that the sign $\sigma(Z)$ of every semicycle Z in S equals to the sign $\sigma(\phi(Z))$ in S'.

Proposition 1.2(E. Sampathkumar et al. [9]) *Two signed digraphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Path Signed Digraphs

In [3], Harary and Norman introduced the notion of line digraphs for digraphs. The *line digraph* $L(D)$ of a given digraph $D = (V, \mathcal{A})$ has the arc set $\mathcal{A} := \mathcal{A}(D)$ of D for its vertex set and (e, f) is an arc in $L(D)$ whenever the arcs e and f in D have a vertex in common in such a way that it is the head of e and the tail of f; hence, a given digraph H is called a *line digraph* if there exists a digraph D such that $L(D) \cong H$. By a natural way, Broersma and Li [1] generalized the concept of line digraphs to that of directed path graphs.

Let k be a positive integer, and denote $\overrightarrow{P_k}$ or $\overrightarrow{C_k}$ a directed path or a directed cycle on k vertices, respectively. Let D be a digraph containing at least one directed path $\overrightarrow{P_k}$. Denote $\Pi_k(D)$, the set of all $\overrightarrow{P_k}$'s of D. Then the *directed $\overrightarrow{P_k}$-graph* of D, denoted by $\overrightarrow{P_k}(D)$, is the digraph with vertex set $\Pi_k(D)$; pq is an arc of $\overrightarrow{P_k}(D)$ if, and only if, there is a $\overrightarrow{P_{k+1}}$ or $\overrightarrow{C_k} = (v_1v_2...v_{k+1})$ in D (with $v_1 = v_{k+1}$ in the case of a $\overrightarrow{C_k}$) such that $p = v_1v_2...v_k$ and

$q = v_2...v_kv_{k+1}$. Note that $\overrightarrow{P_1}(D) = D$ and $\overrightarrow{L}(D)$. In [7], the authors proposed an open problem for further study, i.e., how to give a characterization for directed $\overrightarrow{P_3}$-graphs.

We extend the notion of $\overrightarrow{P_k}(D)$ to the realm of signed digraphs. In a signed digraph $S = (D, \sigma)$, where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. The *path signed digraph* $\overrightarrow{P_k}(S) = (\overrightarrow{P_k}(D), \sigma')$ of a signed digraph $S = (D, \sigma)$ is a signed digraph whose underlying digraph is $\overrightarrow{P_k}(D)$ called *path digraph* and sign of any arc $e = \overrightarrow{P_k}\overrightarrow{P_k'}$ in $\overrightarrow{P_k}(S)$ is $\sigma'(\overrightarrow{P_k}\overrightarrow{P_k'}) = \sigma(\overrightarrow{P_k})\sigma(\overrightarrow{P_k'})$. Further, a signed digraph $S = (G, \sigma)$ is called *path signed digraph*, if $S \cong \overrightarrow{P_k}(S')$, for some signed digraph S'. At the end of this section, we discuss the structural characterization of path signed digraphs $\overrightarrow{P_k}(S)$.We now gives a straightforward, yet interesting, property of path signed digraphs.

Proposition 2.1 *For any signed digraph $S = (D, \sigma)$, its path signed digraph $\overrightarrow{P_k}(S)$ is balanced.*

Proof Since sign of any arc $\sigma'(e = \overrightarrow{P_k}\overrightarrow{P_k'})$ in $\overrightarrow{P_k}(S)$ is $\sigma(\overrightarrow{P_k})\sigma(\overrightarrow{P_k'})$, where σ is the marking of $\overrightarrow{P_k}(S)$, by Proposition 1.1, $\overrightarrow{P_k}(S)$ is balanced. □

Remark: For any two signed digraphs S and S' with same underlying digraph, their path signed digraphs are switching equivalent.

In [9], the authors defined line signed digraph of a signed digraph $S = (D, \sigma)$ as follows:

A line signed digraph $L(S)$ of a signed digraph $S = (D, \sigma)$ is a signed digraph $L(S) = (L(D), \sigma')$ where for any arc $\overrightarrow{ee'}$ in $L(D)$, $\sigma'(\overrightarrow{ee'}) = \sigma(\overrightarrow{e})\sigma(\overrightarrow{e'})$ (see also, E. Sampathkumar et al. [8]).

Hence, we shall call a given signed digraph S a *line signed digraph* if it is isomorphic to the line signed digraph $L(S')$ of some signed digraph S'. By the definition of path signed digraphs, we observe that $\overrightarrow{P_2}(S) = L(S)$.

Corollary 2.2 *For any signed digraph $S = (G, \sigma)$, its $\overrightarrow{P_2}(S)$ (=$L(S)$) is balanced.*

In [9], the authors obtain structural characterization of line signed digraphs as follows:

Proposition 2.3(E. Sampathkumar et al. [9]) *A signed digraph $S = (D, \sigma)$ is a line signed digraph (or $\overrightarrow{P_2}$-signed digraph) if, and only if, S is balanced signed digraph and its underlying digraph D is a line digraph (or $\overrightarrow{P_2}$-digraph).*

Proof Suppose that S is balanced and D is a line digraph. Then there exists a digraph D' such that $L(D') \cong D$. Since S is balanced, by Proposition 1.1, there exists a marking μ of D such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$. Now consider the signed digraph $S' = (D', \sigma')$, where for any arc $\overrightarrow{e}$ in D', $\sigma'(\overrightarrow{e})$ is the marking of the corresponding vertex in D. Then clearly, $L(S') \cong S$. Hence S is a line signed digraph.

Conversely, suppose that $S = (D, \sigma)$ is a line signed digraph. Then there exists a signed digraph $S' = (D', \sigma')$ such that $L(S') \cong S$. Hence D is the line digraph of D' and by Corollary 2.2, S is balanced. □

We strongly believe that the above Proposition can be generalized to path signed digraphs

$\overrightarrow{P_k}(S)$ for $k \geq 3$. Hence, we pose it as a problem:

Problem 2.4 *If $S = (D, \sigma)$ is a balanced signed digraph and its underlying digraph D is a path digraph, then S is a path signed digraph.*

§3. Switching Equivalence of Signed Digraphs and Path Signed Digraphs

Broersma and Li [1] concluded that the only connected digraphs D with $\overrightarrow{P_3}(D) \cong D$ consists of a directed cycle with in-trees or out-trees attached to its vertices, with at most non-trivial trees, where a directed tree T of D is an *out-tree* of D if $V(T) = V(D)$ and precisely one vertex of T has in-degree zero (the root of T), while all other vertices of T have in-degree one, and an *in-tree* of D is defined analogously with respect to out-degrees.

Proposition 3.1(Broersma and Hoede [1]) *Let D be connected digraph without sources or sinks. If D has an in-tree or out-tree, then $\overrightarrow{P_3}(D) \cong D$ if, and only if, $D \cong \overrightarrow{C_n}$ for some $n \geq 3$. Hence, if D is strongly connected, then $\overrightarrow{P_3}(D) \cong D$ if, and only if, $D \cong \overrightarrow{C_n}$ for some $n \geq 3$.*

In the view of the above result, we now characterize signed digraphs that are switching equivalent to their $\overrightarrow{P_3}$-signed digraphs.

Proposition 3.2 *For any strongly connected signed digraph $S = (D, \sigma)$, $S \sim \overrightarrow{P_3}(S)$ if, and only if, S is balanced and $D \cong \overrightarrow{C_n}$ for some $n \geq 3$.*vskip 3mm

Proof Suppose $S \sim L(S)$. This implies, $D \cong L(D)$ and hence by Proposition 3.1, $D \cong \overrightarrow{C_n}$. Now, if S is signed digraph, then by Corollary 2.2, implies that $L(S)$ is balanced and hence if S is unbalanced its line signed digraph $L(S)$ being balanced cannot be switching equivalent to S in accordance with Proposition 1.2. Therefore, S must be balanced.

Suppose that S is balanced and $D \cong \overrightarrow{C_n}$ for some $n \geq 3$. Then, by Proposition 2.1, $\overrightarrow{P_3}(S)$ is balanced, the result follows from Proposition 1.2. $\square$

In [9], the authors defined a signed digraph S is *periodic*, if $L^{n+k}(S) \sim L^n(S)$ for some positive integers n and k.

Analogous to the line signed digraphs, we defined periodic for $\overrightarrow{P_3}(S)$ as follows:

For some positive integers n and k, define that a path signed digraph $\overrightarrow{P_3}(S)$ is periodic, if $\overrightarrow{P_3^{n+k}}(S) \sim \overrightarrow{P_3^n}(S)$.

Proposition 3.3(Broersma and Hoede [1]) *If D is strongly connected digraph and $\overrightarrow{P_3^n}(D) \cong D$ for some $n \geq 1$, then $\overrightarrow{P_3}(D) \cong D$ and D is a directed cycle.*

The following result is follows from Propositions 2.1,3.2 and 3.3.

Proposition 3.4 *If S is strongly connected signed digraph, and $\overrightarrow{P_3^n}(S) \sim S$ for some $n \geq 1$, then $\overrightarrow{P_3}(S) \sim S$ and D is a directed cycle.*

The *negation* $\eta(S)$ of a given signed digraph S defined as follows: $\eta(S)$ has the same underlying digraph as that of S with the sign of each arc opposite to that given to it in S.

However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed digraph $S = (D, \sigma)$, the $\overrightarrow{P_k}(S)$ is balanced (Proposition 2.1). We now examine, the condition under which negation of $\overrightarrow{P_k}(S)$ (i.e., $\eta(\overrightarrow{P_k}(S))$) is balanced.

Proposition 3.5 *Let $S = (D, \sigma)$ be a signed digraph. If $\overrightarrow{P_k}(D)$ is bipartite then $\eta(\overrightarrow{P_k}(S))$ is balanced.*

proof Since, by Proposition 2.1, $\overrightarrow{P_k}(S)$ is balanced, then every semicycle in $\overrightarrow{P_k}(S)$ contains even number of negative arcs. Also, since $\overrightarrow{P_k}(G)$ is bipartite, all semicycles have even length; thus, the number of positive arcs on any semicycle C in $\overrightarrow{P_k}(S)$ are also even. This implies that the same thing is true in negation of $\overrightarrow{P_k}(S)$. Hence $\eta(\overrightarrow{P_k}(S))$ is balanced. □

Proposition 3.2 provides easy solutions to three other signed digraph switching equivalence relations, which are given in the following results.

Corollary 3.6 *For any signed digraph $S = (D, \sigma)$, $\eta(S) \sim \overrightarrow{P_3}(S)$ if, and only if, S is an unbalanced signed digraph on any odd semicycle.*

Corollary 3.7 *For any signed digraph $S = (D, \sigma)$ and for any integer $k \geq 1$, $\overrightarrow{P_k}(\eta(S)) \sim \overrightarrow{P_k}(S)$.*

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