The $t$-Pebbling Number of Jahangir Graph

A.Lourdusamy
(Department of Mathematics, St.Xavier’s College (Autonomous), Palayamkottai - 627 002, India)

S.Samuel Jeyaseelan
(Department of Mathematics, Loyola College (Autonomous),Chennai - 600 002, India)

T.Mathivanan
(Department of Mathematics, St.Xavier's College (Autonomous), Palayamkottai - 627 002, India)

E-mail: lourdugnanam@hotmail.com, samjeya@yahoo.com, tahitvanan@yahoo.com

Abstract: Given a configuration of pebbles on the vertices of a connected graph $G$, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The $t$-pebbling number, $f_t(G)$ of a graph $G$ is the least number $m$ such that, however $m$ pebbles are placed on the vertices of $G$, we can move $t$ pebbles to any vertex by a sequence of pebbling moves. In this paper, we determine $f_t(G)$ for Jahangir graph $J_{2,m}$.

Key Words: Smarandachely $d$-pebbling move, Smarandachely $d$-pebbling number, pebbling move, $t$-pebbling number, Jahangir graph.

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§1. Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [2]. There have been many developments since Hulbert’s survey appeared.

Given a graph $G$, distribute $k$ pebbles (indistinguishable markers) on its vertices in some configuration $C$. Specifically, a configuration on a graph $G$ is a function from $V(G)$ to $N \cup \{0\}$ representing an arrangement of pebbles on $G$. For our purposes, we will always assume that $G$ is connected. A Smarandachely $d$-pebbling move (Smarandachely $d$-pebbling step) is defined as the removal of two pebbles from some vertex and the replacement of one of these pebbles on such a vertex with distance $d$ to the initial vertex with pebbles and the Smarandachely $(t,d)$-pebbling number $f_t^d(G)$, is defined to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex $v$, $t$ pebbles by a sequence.

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of Smarandachely d-pebbling moves. Particularly, if \( d = 1 \), such a Smarandachely 1-pebbling move is called a pebbling move (or pebbling step) and the Smarandache \((t,1)\)-pebbling number \( f_t^1(G) \) is abbreviated to \( f_t(G) \), i.e., it is possible to move to any root vertex \( v \), \( t \) pebbles by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex \( v \) one desires to move to another root vertex, the pebbles reset to their original configuration. There are certain results regarding the \( t \)-pebbling graphs that are investigated in [3-6,9].

**Definition 1.1** Jahangir graph \( J_{n,m} \) for \( m \geq 3 \) is a graph on \( nm + 1 \) vertices, that is, a graph consisting of a cycle \( C_{nm} \) with one additional vertex which is adjacent to \( C_{nm} \).

**Example 1.2** Fig.1 shows Jahangir graph \( J_{2,8} \). The graph \( J_{2,8} \) appears on Jahangir’s tomb in his mausoleum. It lies in 5 kilometer north-west of Lahore, Pakistan, across the River Ravi.

![Fig.1 J_{2,8}](image)

**Remark 1.3** Let \( v_{2m+1} \) be the label of the center vertex and \( v_1, v_2, \cdots, v_{2m} \) be the label of the vertices that are incident clockwise on cycle \( C_{2m} \) so that \( \deg(v_1) = 3 \).

In Section 2, we determine the \( t \)-pebbling number for Jahangir graph \( J_{2,m} \). For that we use the following theorems.

**Theorem 1.4([7])** For the Jahangir graph \( J_{2,3} \), \( f(J_{2,3}) = 8 \).

**Theorem 1.5([7])** For the Jahangir graph \( J_{2,4} \), \( f(J_{2,4}) = 16 \).

**Theorem 1.6([7])** For the Jahangir graph \( J_{2,5} \), \( f(J_{2,5}) = 18 \).

**Theorem 1.7([7])** For the Jahangir graph \( J_{2,6} \), \( f(J_{2,6}) = 21 \).

**Theorem 1.8([7])** For the Jahangir graph \( J_{2,7} \), \( f(J_{2,7}) = 23 \).

**Theorem 1.9([8])** For the Jahangir graph \( J_{2,m}(m \geq 8) \), \( f(J_{2,m}) = 2m + 10 \).

We now proceed to find the \( t \)-pebbling number for \( J_{2,m} \).
§2. The $t$-Pebbling Number for Jahangir Graph $J_{2,m}, m \geq 3$

**Theorem 2.1** For the Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.

*Proof* Consider the Jahangir graph $J_{2,3}$. We prove this theorem by induction on $t$. By Theorem 1.4, the result is true for $t = 1$. For $t > 1$, $J_{2,3}$ contains at least 16 pebbles. Using at most 8 pebbles, we can put a pebble on any desired vertex, say $v_i (1 \leq i \leq 7)$, by Theorem 1.4. Then, the remaining number of pebbles on the vertices of $J_{2,3}$ is at least $8t - 8$. By induction we can put $t - 1$ additional pebbles on the desired vertex $v_i (1 \leq i \leq 7)$. So, the result is true for all $t$. Thus, $f_t(J_{2,3}) \leq 8t$.

Now, consider the following configuration $C$ such that $C(v_4) = 8t - 1$, and $C(x) = 0$, where $x \in V \setminus \{v_4\}$, then we cannot move $t$ pebbles to the vertex $v_1$. Thus, $f_t(J_{2,3}) \geq 8t$. Therefore, $f_t(J_{2,3}) = 8t$. $lacksquare$

**Theorem 2.2** For the Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.

*Proof* Consider the Jahangir graph $J_{2,4}$. We prove this theorem by induction on $t$. By Theorem 1.5, the result is true for $t = 1$. For $t > 1$, $J_{2,4}$ contains at least 32 pebbles. By Theorem 1.5, using at most 16 pebbles, we can put a pebble on any desired vertex, say $v_i (1 \leq i \leq 9)$. Then, the remaining number of pebbles on the vertices of $J_{2,4}$ is at least $16t - 16$. By induction, we can put $t - 1$ additional pebbles on the desired vertex $v_i (1 \leq i \leq 9)$. So, the result is true for all $t$. Thus, $f_t(J_{2,4}) \leq 16t$.

Now, consider the following configuration $C$ such that $C(v_6) = 16t - 1$, and $C(x) = 0$, where $x \in V \setminus \{v_6\}$, then we cannot move $t$ pebbles to the vertex $v_2$. Thus, $f_t(J_{2,4}) \geq 16t$. Therefore, $f_t(J_{2,4}) = 16t$. $lacksquare$

**Theorem 2.3** For the Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.

*Proof* Consider the Jahangir graph $J_{2,5}$. We prove this theorem by induction on $t$. By Theorem 1.6, the result is true for $t = 1$. For $t > 1$, $J_{2,5}$ contains at least 34 pebbles. Using at most 16 pebbles, we can put a pebble on any desired vertex, say $v_i (1 \leq i \leq 11)$. Then, the remaining number of pebbles on the vertices of the graph $J_{2,5}$ is at least $16t - 14$. By induction, we can put $t - 1$ additional pebbles on the desired vertex $v_i (1 \leq i \leq 11)$. So, the result is true for all $t$. Thus, $f_t(J_{2,5}) \leq 16t + 2$.

Now, consider the following distribution $C$ such that $C(v_6) = 16t - 1, C(v_8) = 1, C(v_{10}) = 1$ and $C(x) = 0$, where $x \in V \setminus \{v_6, v_8, v_{10}\}$. Then we cannot move $t$ pebbles to the vertex $v_2$. Thus, $f_t(J_{2,5}) \geq 16t + 2$. Therefore, $f_t(J_{2,5}) = 16t + 2$. $lacksquare$

**Theorem 2.4** For the Jahangir graph $J_{2,m} (m \geq 6)$, $f_t(J_{2,m}) = 16(t - 1) + f(J_{2,m})$.

*Proof* Consider the Jahangir graph $J_{2,m}$, where $m > 5$. We prove this theorem by induction on $t$. By Theorems 1.7 – 1.9, the result is true for $t = 1$. For $t > 1$, $J_{2,m}$ contains at least

$$16 + f(J_{2,m}) = 16 + \begin{cases} 2m + 9 & m = 6, 7 \\ 2m + 10 & m \geq 8. \end{cases}$$

pebbles. Using at most 16 pebbles, we can put a
pebble on any desired vertex, say \( v_i (1 \leq i \leq 2m + 1) \). Then, the remaining number of pebbles
on the vertices of the graph \( J_{2,m} \) is at least \( 16t + f(J_{2,m}) - 32 \). By induction, we can put \( t - 1 \)
additional pebbles on the desired vertex \( v_i (1 \leq i \leq 2m + 1) \). So, the result is true for all \( t \).
Thus, \( f_t(J_{2,m}) \leq 16(t - 1) + f(J_{2,m}) \).

Now, consider the following distributions on the vertices of \( J_{2,m} \).

For \( m = 6 \), consider the following distribution \( C \) such that \( C(v_6) = 16(t - 1) + 15, C(v_{10}) = 3, C(v_8) = 1, C(v_{12}) = 1 \) and \( C(x) = 0 \), where \( x \in V \setminus \{v_6, v_8, v_{10}, v_{12}\} \).

For \( m = 7 \), consider the following distribution \( C \) such that \( C(v_6) = 16(t - 1) + 15, C(v_{10}) = 3, C(v_8) = C(v_{12}) = C(v_{13}) = C(v_{14}) = 1 \) and \( C(x) = 0 \), where \( x \in V \setminus \{v_6, v_8, v_{10}, v_{12}, v_{13}, v_{14}\} \).

For \( m \geq 8 \), if \( m \) is even, consider the following distribution \( C_1 \) such that \( C_1(v_{m+2}) = 16(t - 1) + 15, C_1(v_{m-2}) = 3, C_1(v_{m+6}) = 3, C_1(x) = 1 \), where \( x \in \{N[v_2], N[v_{m+2}], N[v_{m-2}], N[v_{m+6}]\} \)
and \( C_1(y) = 0 \), where \( y \in \{N[v_2], N(v_{m+2}), N(v_{m-2}), N(v_{m+6})\} \).

If \( m \) is odd, then consider the following configuration \( C_2 \) such that \( C_2(v_{m+1}) = 16(t - 1) + 15, C_2(v_{m-3}) = 3, C_2(v_{m+5}) = 3, C_2(x) = 1 \), where \( x \in \{N[v_2], N[v_{m+1}], N[v_{m-3}], N[v_{m+5}]\} \)
and \( C_2(y) = 0 \), where \( y \in \{N[v_2], N(v_{m+1}), N(v_{m-3}), N(v_{m+5})\} \). Then, we cannot move \( t \) pebbles
to the vertex \( v_2 \) of \( J_{2,m} \) for all \( m \geq 6 \). Thus, \( f_t(J_{2,m}) \geq 16(t - 1) + f(J_{2,m}) \). Therefore, \( f_t(J_{2,m}) = 16(t - 1) + f(J_{2,m}) \). \( \square \)

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