ON THE M-TH POWER RESIDUE OF N *

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Abstract    For any positive integer n, let \( a_m(n) \) denote the m-th power residue of n. In this paper, we use the elementary method to study the asymptotic properties of \( \log (a_m(n!)) \), and give an interesting asymptotic formula for it.

Keywords: m-th power residue of n; Chebyshev’s function; Asymptotic formula.

§1. Introduction

Let \( m > 2 \) be a fixed integer. For any positive integer n, we define \( a_m(n) \) as the m-th power residue of n (See reference [1]). That is, if \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) denotes the factorization of n into prime powers, then \( a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r} \), where \( \beta_i = \min(\alpha_i, m-1) \). Let p be a prime, and for any real number \( x > 1 \), \( \theta(x) = \sum_{p \leq x} \log p \) denotes the Chebyshev’s function of x. In this paper, we will use the elementary methods to study the asymptotic properties of \( \log (a_m(n!)) \), and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. Let \( m > 1 \) be a fixed positive integer. Then for any positive integer n, we have the asymptotic formula:

\[
\log (a_m(n!)) = n \left( \sum_{a=1}^{m-1} \frac{1}{a} \right) + O \left( n \exp \left( \frac{-A \log^2 n}{(\log \log n)^2} \right) \right),
\]

where \( A \) is a fixed positive constant.

§2. Proof of the theorem

Before the proof of Theorem, a lemma will be useful.

*This work is supported by the N.S.F.(10271093) of P.R.China and the Education Department Foundation of Shaanxi Province(04JK132).
Lemma. Let \( p \) be a prime. Then for any real number \( x \geq 2 \), we have the asymptotic formula:

\[
\theta(x) = x + O\left(x \exp \left(-A \frac{\log^2 x}{\log \log x}\right)\right),
\]

where \( A \) is a positive constant.


Now we use this Lemma to complete the proof of Theorem. In fact, let \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) denotes the factorization of \( n \) into prime powers. Suppose that \( m \ll n \), if \( (m-1)p \leq n < mp \), then \( p^{m-1} \parallel n \!). From the definition of \( a_m(n) \), we can write

\[
a_m(n!) = \prod_{\frac{n}{p} < p \leq n} p \prod_{\frac{n}{q} < q \leq \frac{n}{p}} p^2 \cdots \prod_{\alpha_{m-1} < p \leq \frac{n}{m-2}} p^{m-2} \prod_{p \leq \frac{n}{m-1}} p^{m-1}.
\]

By taking the logistic computation in the two sides, we have

\[
\log(a_m(n!)) = \theta(n) - \theta\left(\frac{n}{2}\right) + 2\left(\theta\left(\frac{n}{3}\right) - \theta\left(\frac{n}{2}\right)\right) + \cdots + (m-2)\left(\theta\left(\frac{n}{m-2}\right) - \theta\left(\frac{n}{m-1}\right)\right) + (m-1)\theta\left(\frac{n}{m-1}\right)
\]

Then, combining Lemma, we can get the asymptotic formula:

\[
\log(a_m(n!)) = n + \frac{n}{2} + \cdots + \frac{n}{m-1} + O\left(n \exp \left(-A \frac{\log^2 n}{\log \log n}\right)\right)
\]

This completes the proof of Theorem.

References

On the \( m \)-th power residue of \( n \)
