Smarandachely Precontinuous maps

and Preopen Sets in Topological Vector Spaces

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Abstract: It is shown that linear functional on topological vector spaces are Smarandachely precontinuous. Prebounded, totally prebounded and precompact sets in topological vector spaces are identified.

Key Words: Smarandachely Preopen set, precompact set, Smarandachely precontinuous map.

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§1. Introduction

N. Levine [7] introduced the theory of semi-open sets and the theory of α-sets for topological spaces. For a systematic development of semi-open sets and the theory of α-sets one may refer to [1], [2], [4], [5] and [9]. The notion of preopen sets for topological spaces was introduced by S. N. Mashour, M. E. Abd El-Moncef and S.N. El-Deep in [8]. These concepts above are closely related. It is known that, in a topological space, a set is preopen and semi-open if and only if it is an α-set [10], [11]. Our object in section 3 is to define a prebounded set, totally prebounded set, and precompact set in a topological vector space. In Sections 3 and 4 we identify them. Moreover, in Section 2, we show that every linear functional on a topological vector space is precontinuous and deduce that every topological vector space is a prehausdorff space.

§2. Precontinuous maps

We recall the following definitions [2], [8].

Definition 2.1 Let $X$ be a topological space. A subset $S$ of $X$ is said to be Smarandachely preopen if there exists a set $U \subseteq \text{cl}(S)$ such that $S \subseteq \text{int}((\text{cl}(S) \cup U))$. A Smarandachely preneighbourhood of the point $x \in X$ is any Smarandachely preopen set containing $x$. Particularly, a

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Smarandachely \( \emptyset \)-preopen set \( S \) is usually called a preopen set.

**Definition 2.2** Let \( X \) and \( Y \) be topological spaces and \( f : X \to Y \). The function \( f \) is said to be Smarandachely precontinuous if the inverse image \( f^{-1}(B) \) of each open set \( B \) in \( Y \) is a Smarandachely preopen set in \( X \). The function \( f \) is said to be Smarandachely preopen if the image \( f(A) \) of every open set \( A \) in \( X \) is Smarandachely preopen in \( Y \). Particularly, if we replace each Smarandachely preopen by preopen, \( f \) is called to be precontinuous.

The following lemma is obvious.

**Lemma 2.1** Let \( X \) and \( Y \) be topological vector spaces and \( f : X \to Y \) linear. The function \( f \) is preopen if and only if, for every open set \( U \) containing \( 0 \in X \), \( 0 \in Y \) is an interior point of \( \text{cl}(f(U)) \).

The following two theorems are known but we include the proofs for convenience of the reader.

**Theorem 2.1** Let \( X,Y \) be topological vector spaces and let \( Y \) have the Baire property, that is, whenever \( Y = \bigcup_{n=1}^{\infty} B_n \) with closed sets \( B_n \), there is is \( N \) such that \( \text{int}(B_N) \) is nonempty. Let \( f : X \to Y \) be linear and \( f(X) = Y \). Then \( f \) is preopen.

*Proof* Let \( U \subset X \) be a neighborhood of 0. There is a neighborhood \( V \) of 0 such that \( V - V \subset U \). Since \( V \) is a neighborhood of 0 we have \( X = \bigcup_{n=1}^{\infty} nV \). It follows from linearity and surjectivity of \( f \) that \( Y = \bigcup_{n=1}^{\infty} nf(V) \). Since \( Y \) has the Baire property, there is \( N \) such that \( \text{cl}(Nf(V)) = N\text{cl}(f(V)) \) contains an open set \( S \) which is not empty. Then \( \text{cl}(f(V)) \) contains the open set \( T = \frac{1}{N}S \). It follows that

\[
T - T \subset \text{cl}(f(V)) - \text{cl}(f(V)) \subset \text{cl}(f(V) - f(V)) = \text{cl}(f(V - V)) \subset \text{cl}(f(U)).
\]

The set \( T - T \) is open and contains 0. Therefore, \( 0 \in Y \) is an interior point of \( \text{cl}(f(U)) \). From Lemma 2.1 we conclude that \( f \) is preopen. \( \square \)

Note that \( f \) can be any linear surjective map. It is not necessary to assume that \( f \) is continuous or precontinuous.

**Theorem 2.2** Let \( X,Y \) be topological vector spaces, and let \( X \) have the Baire property. Then every linear map \( f : X \to Y \) is precontinuous.

*Proof* Let \( G = \{(x,f(x)) : x \in X \} \) be the graph of \( f \). The projections \( \pi_1 : G \to X \) and \( \pi_2 : G \to Y \) are continuous. The projection \( \pi_1 : G \to X \) is bijective. It follows from Theorem ?? that \( \pi_1 \) is preopen. Therefore, the inverse mapping \( \pi_1^{-1} \) is precontinuous. Then \( f = \pi_2 \circ \pi_1^{-1} \) is precontinuous. \( \square \)

Theorem 2.2 shows that many linear maps are automatically precontinuous. Therefore, it is natural to ask for an example of a linear map which is not precontinuous.

Let \( X = C[0,1] \) be the vector space of real-valued continuous functions on \([0,1]\) equipped with the norm

\[
\|f\| = \max_{x \in [0,1]} |f(x)|.
\]
\[ \|f\|_1 = \int_0^1 |f(x)| \, dx. \]

Let \( Y = C[0,1] \) be equipped with the norm
\[ \|f\|_\infty = \max_{x \in [0,1]} |f(x)|. \]

**Lemma 2.2** The identity operator \( T : X \to Y \) is not precontinuous.

**Proof** Let \( U = \{ f \in C[0,1] : \|f\|_\infty < 1 \} \) which is an open subset of \( Y \). Let \( \text{cl}(U) \) be the closure of \( U \) in \( X \). We claim that
\[ (2.1) \quad \text{cl}(U) \subset \{ f \in C[0,1] : \|f\|_\infty \leq 1 \}. \]

For the proof, consider a sequence \( f_n \in U \) and a function \( f \in C[0,1] \) such that \( \{ f_n \} \) converges to \( f \) in \( X \). Suppose that there is \( x_0 \in [0,1] \) such that \( f(x_0) > 1 \). By continuity of \( f \), there are \( a < b \) and \( \delta > 0 \) such that \( 0 \leq a \leq x_0 \leq b \leq 1 \) and \( f(x) > 1 + \delta \) for \( x \in (a, b) \). Then, as \( n \to \infty \),
\[ (b - a)\delta \leq \int_a^b |f_n(x) - f(x)| \, dx \leq \int_0^1 |f_n(x) - f(x)| \, dx \to 0 \]
which is a contradiction. Therefore, \( f(x) \leq 1 \) for all \( x \in [0,1] \). Similarly, we show that \( f(x) \geq -1 \) for all \( x \in [0,1] \). Now \( 0 \in U = T^{-1}(U) \) but \( U \) is not preopen in \( X \). We see this as follows. Suppose that \( U \) is preopen in \( X \). The sequence \( g_n(x) = 2x^n \) converges to 0 in \( X \). Therefore, \( g_n \in \text{cl}(U) \) for some \( n \) and (2.1) implies \( 2 = \|g_n\|_\infty \leq 1 \) which is a contradiction. □

We can improve Theorem 2.2 for linear functionals.

**Theorem 2.3** Let \( f \) be a linear functional on a topological vector space \( X \). If \( V \) is a preopen subset of \( \mathbb{R} \) then \( f^{-1}(V) \) is a preopen subset of \( X \). In particular, \( f \) is precontinuous.

**Proof** We distinguish the cases that \( f \) is continuous or discontinuous.

Suppose that \( f \) is continuous. If \( f(x) = 0 \) for all \( x \in X \) the statement of the theorem is true. Suppose that \( f \) is onto. We choose \( u \in X \) such that \( f(u) = 1 \). Let \( V \) be a preopen subset of \( \mathbb{R} \), and set \( U := f^{-1}(V) \). Let \( x \in U \) so \( f(x) \in V \). Since \( V \) is preopen, there is \( \delta > 0 \) such that
\[ (2.2) \quad I := (f(x) - \delta, f(x) + \delta) \subset \text{cl}(V). \]

Since \( f \) is continuous, \( f^{-1}(I) \) is an open subset of \( X \) containing \( x \). We claim that
\[ (2.3) \quad f^{-1}(I) \subset \text{cl}(U). \]

In order to prove (2.3), let \( y \in f^{-1}(I) \) so \( f(y) \in I \). By (2.2), there is a sequence \( \{t_n\} \) in \( V \) converging to \( f(y) \). Set
\[ y_n := y + (t_n - f(y))u. \]
We have $f(y_n) = t_n \in V$ so $y_n \in U$. Since $X$ is a topological vector space, $y_n$ converges to $y$. This establishes (2.3). It follows that $U$ is preopen.

Suppose now that $f$ is not continuous. By [3, Corollary 22.1], $N(f) = \{x \in X : f(x) = 0\}$ is not closed. Therefore, there is $y \in \text{cl}(N(f))$ such that $y \notin N(f)$ so $f(y) \neq 0$. Let $x$ be any vector in $X$. There is $t \in \mathbb{R}$ such that $f(x) = tf(y)$ and so $x - ty \in N(f)$. It follows that $x \in \text{cl}(N(f))$. We have shown that $N(f)$ is dense in $X$. Let $a \in \mathbb{R}$. There is $y \in X$ such that $f(y) = a$. Then $f^{-1}(\{a\}) = y + N(f)$ and so the closure of $f^{-1}(\{a\})$ is $y + \text{cl}(N(f)) = X$. Therefore, $f^{-1}(\{a\})$ is dense for every $a \in \mathbb{R}$. Let $V$ be a preopen set in $\mathbb{R}$. If $V$ is empty then $f^{-1}(V)$ is empty and so is preopen. If $V$ is not empty choose $a \in V$. Then $f^{-1}(V) \supset f^{-1}(\{a\})$ and so $f^{-1}(V)$ is dense. Therefore, $f^{-1}(V)$ is preopen. □

§3. Subsets of topological vector spaces

In this section our principal goal is to define prebounded sets, totally prebounded sets and precompact sets in a topological vector space, and to find relations between them. We begin this section with some definitions.

**Definition 3.1** A subset $E$ of a topological vector space $X$ is said to be prebounded if for every preneighbourhood $V$ of $0$ there exists $s > 0$ such that $E \subset tV$ for all $t > s$.

**Definition 3.2** A subset $E$ of a topological vector space $X$ is said to be totally prebounded if for every preneighbourhood $U$ of $0$ there exists a finite subset $F$ of $X$ such that $E \subset F + U$.

**Definition 3.3** A subset $E$ of a topological vector space $X$ is said to be precompact if every preopen cover of $E$ admits a finite subcover.

**Lemma 3.1** Every precompact set in a topological vector space $X$ is totally prebounded.

**Proof** Let $E$ be precompact. Let $V$ be preopen with $0 \in V$. Then the collection $\{x + V : x \in E\}$ is a cover of $E$ consisting of preopen sets. There are $x_1, x_2, \ldots, x_n \in E$ such that $E \subset \bigcup_{i=1}^{n} \{x_i + V\}$. Therefore, $E$ is totally prebounded. □

**Lemma 3.2** In a topological vector space $X$ the singleton $\{0\}$ is the only prebounded set.

**Proof** It is enough to show that every singleton $\{u\}$, $u \neq 0$, is not prebounded. Let $V = X - \{\frac{1}{n}u : n \in \mathbb{N}\}$. The closure of $V$ is $X$ so $V$ is preopen. But $\{u\}$ is not subset of $nV$ for $n = 1, 2, 3, \ldots$. Therefore, $\{u\}$ is not prebounded. □

**Theorem 3.1** If $E$ is a prebounded subset of a topological vector space $X$, then $E$ is totally prebounded. The converse statement is not true.

**Proof** This follows from the fact that every finite set is totally prebounded, and by using Lemma 3.2. □
§4. Applications of Theorem 2.3

We need the following known lemma.

Lemma 4.1 If $U, V$ are two vector spaces, and $W$ is a linear subspace of $U$ and $f : W \to V$ is a linear map, then there is a linear map $g : U \to V$ such that $f(x) = g(x)$ for all $x \in W$.

Proof We choose a basis $A$ in $W$ and then extend to a basis $B \supset A$ in $U$. We define $h(a) = f(a)$ for $a \in A$ and $h(b)$ arbitrary in $V$ for $b \in B - A$. There is a unique linear map $g : U \to V$ such that $g(b) = h(b)$ for $b \in B$. Then $g(x) = f(x)$ for all $x \in W$. □

We obtain the following result.

Theorem 4.1 Every topological vector space $X$ is a prehausdorff space, that is, for each $x, y \in X$, $x \neq y$, there exists a preneighbourhood $U$ of $x$ and a preneighbourhood $V$ of $y$ such that $U \cap V = \emptyset$.

Proof Let $x, y \in X$ and $x \neq y$. If $x, y$ are linearly dependent we choose a linear functional on the span of $\{x, y\}$ such that $f(x) < f(y)$. If $x, y$ are linearly independent we set $f(sx + ty) = s$. By Lemma 4.1 we extend $f$ to a linear functional $g$ with $g(x) < g(y)$. Choose $c \in (g(x), g(y))$ and define $U = g^{-1}((c, \infty))$ and $V = g^{-1}((-\infty, c))$. Then, using Theorem 2.3, $U, V$ are preopen. Also $U$ and $V$ are disjoint and $x \in U$, $y \in V$. □

We now determine totally prebounded subsets in $\mathbb{R}$. The result may not be surprising but the proof requires some care.

Lemma 4.2 A subset of $\mathbb{R}$ is totally prebounded if and only if it is finite.

Proof It is clear that a finite set is totally prebounded. Let $E$ be a countable (finite or infinite) subset of $\mathbb{R}$ which is totally prebounded. Let $A := \{x - y : x, y \in E\}$. The set $A$ is countable. We define a sequence $\{u_n\}$ of real numbers inductively as follows. We set $u_1 = 0$. Then we choose $u_2 \in (-1, 0)$ such that $u_2 - u_1 \notin A$. Then we choose $u_3 \in (0, 1)$ such that $u_3 - u_1 \notin A$ for $i = 1, 2$. Then we choose $u_4 \in (-1, -\frac{1}{2})$ such that $u_4 - u_2 \notin A$ for $i = 1, 2, 3$. Continuing in this way we construct a set $U = \{u_n : n \in \mathbb{N}\} \subset (-1, 1)$ such that every interval of the form $(m2^{-k}, (m + 1)2^{-k})$ with $-2^k < m < 2^k$, $k \in \mathbb{N}$, contains at least one element of $U$, and such that $0 \in U$ and $u - v \notin A$ for all $u, v \in U$, $u \neq v$. Then $cl(U) = [-1, 1]$ so $U$ is a preneighbourhood of 0. Since $E$ is totally prebounded, there is a finite set $F$ such that $E \subset F + U$. If $z \in F$ and $x, y \in E$ lie in $z + U$ then $x = z + u$, $y = z + v$ with $u, v \in U$. It follows that $u - v = x - y \notin A$ and, by construction of $U$, $u = v$. Therefore, $x = y$ and so each set $z + U$, $z \in F$, contains at most one element of $E$. Therefore, $E$ is finite. We have shown that every countable set which is totally prebounded is finite. It follows that every totally prebounded set is finite. □

Combining several of our results we can now identify totally prebounded and precompact subset of any topological vector space.

Theorem 4.2 Let $X$ be a topological vector space. A subset of $X$ is totally prebounded if and only if it is finite. Similarly, a subset of $X$ is precompact if and only if it is finite.

Proof Every finite set is totally prebounded. Conversely, suppose that $E$ is a totally
prebounded subset of $X$. Let $f$ be a linear functional on $X$. It follows easily from Theorem 2.3 that $f(E)$ is a totally prebounded subset of $\mathbb{R}$. By Lemma 4.2, $f(E)$ is finite. It follows that $E$ is finite as we see as follows. Suppose that $E$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ which is linearly independent. Then, using Lemma 4.1, we can construct a linear functional $f$ on $X$ such that $f(x_n) \neq f(x_m)$ if $n \neq m$. This is a contradiction so $E$ must lie in a finite dimensional subspace $Y$ of $X$. We choose a basis $y_1, \ldots, y_k$ in $Y$, and represent each $x \in E$ in this basis 

$$x = f_1(x)y_1 + \cdots + f_k(x)y_k.$$ 

Every $f_j$ is a linear functional on $Y$ so $f_j(E)$ is a finite set for each $j = 1, 2, \ldots, k$. It follows that $E$ is finite.

Clearly, every finite set is precompact. Conversely, by Lemma 3.1, a precompact subset of $X$ is totally prebounded, so it is finite. \qed

References