The product of divisors minimum and maximum functions

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Abstract Let $T(n)$ denote the product of divisors of the positive integer $n$. We introduce and study some basic properties involving two functions, which are the minimum, resp. the maximum of certain integers connected with the divisors of $T(n)$.

Keywords Arithmetic functions, product of divisors of an integer.

1. Let $T(n) = \prod_{i|n} i$ denote the product of all divisors of $n$. The product-of-divisors minimum, resp. maximum functions will be defined by

$$T(n) = \min\{k \geq 1 : n|T(k)\} \quad (1)$$

and

$$T_*(n) = \max\{k \geq 1 : T(k)|n\}. \quad (2)$$

There are particular cases of the functions $F_A^f, G_A^g$ defined by

$$F_A^f(n) = \min\{k \in A : n|f(k)\}, \quad (3)$$

and its "dual"

$$G_A^g(n) = \max\{k \in A : g(k)|n\}, \quad (4)$$

where $A \subset \mathbb{N}^*$ is a given set, and $f, g : \mathbb{N}^* \to \mathbb{N}$ are given functions, introduced in [8] and [9]. For $A = \mathbb{N}^*$, $f(k) = g(k) = k!$ one obtains the Smarandache function $S(n)$, and its dual $S_*(n)$, given by

$$S(n) = \min\{k \geq 1 : n|k!\} \quad (5)$$

and

$$S_*(n) = \max\{k \geq 1 : k!|n\}. \quad (6)$$

The function $S_*(n)$ has been studied in [8], [9], [4], [1], [3]. For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(k)$, one obtains the Euler minimum, resp. maximum functions

$$E(n) = \min\{k \geq 1 : n|\varphi(k)\} \quad (7)$$

studied in [6], [8], [13], resp., its dual

$$E_*(n) = \max\{k \geq 1 : \varphi(k)|n\}, \quad (8)$$
studied in [13].

For \( A = \mathbb{N}^* \), \( f(k) = g(k) = S(k) \) one has the Smarandache minimum and maximum functions

\[
S_{\text{min}}(n) = \min\{ k \geq 1 : n|S(k) \}, \quad (9)
\]

\[
S_{\text{max}}(n) = \max\{ k \geq 1 : S(k)|n \}, \quad (10)
\]

introduced, and studied in [15]. The divisor minimum function

\[
D(n) = \min\{ k \geq 1 : n|d(k) \} \quad (11)
\]

(where \( d(k) \) is the number of divisors of \( k \)) appears in [14], while the sum-of-divisors minimum and maximum functions

\[
\Sigma(n) = \min\{ k \geq 1 : n|\sigma(k) \} \quad (12)
\]

\[
\Sigma^*(n) = \max\{ k \geq 1 : \sigma(k)|n \} \quad (13)
\]

have been recently studied in [16].

For functions \( Q(n), Q_1(n) \) obtained from (3) for \( f(k) = k! \) and \( A = \) set of perfect squares, resp. \( A = \) set of squarefree numbers, see [10].

2. The aim of this note is to study some properties of the functions \( T(n) \) and \( T^*(n) \) given by (1) and (2). We note that properties of \( T(n) \) in connection with "multiplicatively perfect numbers" have been introduced in [11]. For other asymptotic properties of \( T(n) \), see [7]. For divisibility properties of \( T(\sigma(n)) \) with \( T(n) \), see [5]. For asymptotic results of sums of type

\[
\sum_{n \leq x} \frac{1}{T(n)},
\]

see [17].

A divisor \( i \) of \( n \) is called "unitary" if \( (i, \frac{n}{i}) = 1 \). Let \( T^*(n) \) be the product of unitary divisors of \( n \). For similar results to [11] for \( T^*(n) \), or \( T^{**}(n) \) (i.e. the product of "bi-unitary" divisors of \( n \)), see [2]. The product of "exponential" divisors \( T_e(n) \) is introduced in paper [12].

Clearly, one can introduce functions of type (1) and (2) for \( T(n) \) replaced with one of the above functions \( T^*(n), T^{**}, T_e(n) \), but these functions will be studied in another paper.

3. The following auxiliary result will be important in what follows.

**Lemma 1.**

\[
T(n) = n^{d(n)/2},
\]

where \( d(n) \) is the number of divisors of \( n \).

**Proof.** This is well-known, see e.g. [11].

**Lemma 2.**

\[
T(a)|T(b), \quad \text{if } a|b.
\]

**Proof.** If \( a|b \), then for any \( d|a \) one has \( d|b \), so \( T(a)|T(b) \). Reciprocally, if \( T(a)|T(b) \), let \( \gamma_p(a) \) be the exponent of the prime in \( a \). Clearly, if \( p|a \), then \( p|b \), otherwise \( T(a)|T(b) \) is impossible. If \( p^{\gamma_p(b)}|b \), then we must have \( \gamma_p(a) \leq \gamma_p(b) \). Writing this fact for all prime divisors of \( a \), we get \( a|b \).

**Theorem 1.** If \( n \) is squarefree, then

\[
T(n) = n.
\]
Since \( k \) is squarefree (i.e. a product of distinct primes), this implies if \( ab \mid k \), so there is a \( d \mid k \), so that \( p_i \mid d \). But then \( p_i \mid k \) for all \( i = 1, \ldots, r \), thus \( p_1 p_2 \ldots p_r = n \mid k \).

Since \( p_1 p_2 \ldots p_k \mid T(p_1 p_2 \ldots p_k) \), the least \( k \) is exactly \( p_1 p_2 \ldots p_r \), proving (16).

**Remark.** Thus, if \( p \) is a prime, \( T(p) = p ; \) if \( p < q \) are primes, then \( T(pq) = pq \), etc.

**Theorem 2.** If \( a \mid b, a \neq b \) and \( b \) is squarefree, then

\[
T(ab) = b. \tag{17}
\]

**Proof.** If \( a \mid b, a \neq b \), then clearly \( T(b) = \prod_{d \mid b} d \) is divisible by \( ab \), so \( T(ab) \leq b \). Reciprocally, if \( ab \mid T(k) \), let \( p \mid b \) a prime divisor of \( b \). Then \( p \mid T(k) \), so (see the proof of Theorem 1) \( p \mid k \). But \( b \) being squarefree (i.e. a product of distinct primes), this implies \( b \mid k \). The least such \( k \) is clearly \( k = b \).

For example, \( T(12) = T(2 \cdot 6) = 6, T(18) = T(3 \cdot 6) = 6, T(20) = T(2 \cdot 10) = 10 \).

**Theorem 3.** \( T(T(n)) = n \) for all \( n \geq 1 \).

**Proof.** Let \( T(n) \mid T(k) \). Then by (15) one can write \( n \mid k \). The least \( k \) with this property is \( k = n \), proving relation (18).

**Theorem 4.** Let \( p_i (i = 1, \ldots, r) \) be distinct primes, and \( \alpha_i \geq 1 \) positive integers. Then

\[
\max \left\{ T \left( \prod_{i=1}^{r} p_i^{\alpha_i} \right) : i = 1, \ldots, r \right\} \leq T \left( \prod_{i=1}^{r} p_i^{\alpha_i} \right) \leq \text{l.c.m.} \left[ T(p_1^{\alpha_1}), \ldots, T(p_r^{\alpha_r}) \right]. \tag{19}
\]

**Proof.** In [13] it is proved that for \( A = \mathbb{N}^* \), and any function \( f \) such that \( F_f^+ (n) = F_f (n) \) is well defined, one has

\[
\max \{ F_f (p_i^{\alpha_i}) : i = 1, \ldots, r \} \leq F_f \left( \prod_{i=1}^{r} p_i^{\alpha_i} \right). \tag{20}
\]

On the other hand, if \( f \) satisfies the property

\[
a \mid b \implies f(a) \mid f(b) (a, b \geq 1), \tag{21}
\]

then

\[
F_f \left( \prod_{i=1}^{r} p_i^{\alpha_i} \right) \leq \text{l.c.m.} [F_f (p_1^{\alpha_1}), \ldots, F_f (p_r^{\alpha_r})]. \tag{22}
\]

By Lemma 2, (21) is true for \( f(a) = T(a) \), and by using (20), (22), relation (19) follows.

**Theorem 5.**

\[
T(2^n) = 2^n, \tag{23}
\]

where \( \alpha \) is the least positive integer such that

\[
\frac{\alpha (\alpha + 1)}{2} \geq n. \tag{24}
\]

**Proof.** By (14), \( 2^n \mid T(k) \) iff \( 2^n \mid k^{d(k)/2} \). Let \( k = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), when \( d(k) = (\alpha_1 + 1) \cdots (\alpha_r + 1) \).

Since \( 2^{2n} \mid k^{d(k)} = p_1^{\alpha_1(\alpha_1+1)} \cdots p_r^{\alpha_r(\alpha_r+1)} \) (let \( p_1 < p_2 < \cdots < p_r \)), clearly \( p_1 = 2 \)
and the least \( k \) is when \( \alpha_2 = \cdots = \alpha_r = 0 \) and \( \alpha_1 \) is the least positive integer with \( 2n \leq \alpha_1(\alpha_1 + 1) \). This proves (23), with (24).

For example, \( T(2^2) = 4 \), since \( \alpha = 2 \), \( T(2^3) = 4 \) again, \( T(2^4) = 8 \) since \( \alpha = 3 \), etc.

For odd prime powers, the things are more complicated. For example, for \( 3^\alpha \) one has:

**Theorem 6.**

\[
T(3^n) = \min\{3^{\alpha_1}, 2 \cdot 3^{\alpha_2}\},
\]

where \( \alpha_1 \) is the least positive integer such that \( \frac{\alpha_1(\alpha_1 + 1)}{2} \geq n \), and \( \alpha_2 \) is the least positive integer such that \( \alpha_2(\alpha_2 + 1) \geq n \).

**Proof.** As in the proof of Theorem 5,

\[
3^{2n|p_1^{\alpha_1(\alpha_1+1)} \cdots p_r^{\alpha_r(\alpha_r+1)}},
\]

where \( p_1 < p_2 < \cdots < p_r \), so we can distinguish two cases:

a) \( p_1 = 2 \), \( p_2 = 3 \), \( p_3 \geq 5 \);

b) \( p_1 = 3 \), \( p_2 \geq 5 \).

Then \( k = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p_r^{\alpha_r} \geq 2^{\alpha_1} \cdot 3^{\alpha_2} \) in case a), and \( k \geq 3^{\alpha_1} \) in case b). So for the least \( k \) we must have \( \alpha_2(\alpha_1 + 1)(\alpha_2 + 1) \geq 2n \) with \( \alpha_1 = 1 \) in case a), and \( \alpha_1(\alpha_1 + 1) \geq 2n \) in case b). Therefore \( \frac{\alpha_1(\alpha_1 + 1)}{2} \geq n \) and \( \alpha_2(\alpha_2 + 1) \geq n \), and we must select \( k \) with the least of \( 3^{\alpha_1} \) and \( 2^{\alpha_1} \cdot 3^{\alpha_2} \), so Theorem 6 follows.

For example, \( T(3^2) = 6 \) since for \( n = 2 \), \( \alpha_1 = 2 \), \( \alpha_2 = 1 \), and \( \min\{2 \cdot 3^1, 3^2\} = 6 \); \( T(3^3) = 9 \) since for \( n = 3 \), \( \alpha_1 = 2 \), \( \alpha_2 = 2 \) and \( \min\{2 \cdot 3^2, 3^3\} = 9 \).

**Theorem 7.** Let \( f : [1, \infty) \to [0, \infty) \) be given by \( f(x) = \sqrt{x} \log x \). Then

\[
f^{-1}(\log n) < T(n) < n,
\]

for all \( n \geq 1 \), where \( f^{-1} \) denotes the inverse function of \( f \).

**Proof.** Since \( n\sqrt{T(n)} \) the right side of (26) follows by definition (1) of \( T(n) \). On the other hand, by the known inequality \( d(k) < 2\sqrt{k} \), and Lemma 1 (see (14)) we get \( T(k) < k\sqrt{k} \), so \( \log T(k) < \sqrt{k} \log k = f(k) \). Since \( n\sqrt{T(k)} \) implies \( n \leq T(k) \), so \( \log n \leq \log T(k) < f(k) \), and the function \( f \) being strictly increasing and continuous, by the bijectivity of \( f \), the left side of (26) follows.

4. The function \( T_s(n) \) given by (2) differs in many aspects from \( T(n) \). The first such property is:

**Theorem 8.** \( T_s(n) \leq n \) for all \( n \), with equality only if \( n = 1 \) or \( n = \text{prime} \).

**Proof.** If \( T(k)|n \), then \( T(k) \leq n \). But \( T(k) \geq k \), so \( k \leq n \), and the inequality follows.

Let us now suppose that for \( n > 1 \), \( T_s(n) = n \). Then \( T(n)|n \), by definition 2. On the other hand, clearly \( n|T(n) \), so \( T(n) = n \). This is possible only when \( n = \text{prime} \).

**Remark.** Therefore the equality

\[
T_s(n) = n(n > 1)
\]

is a characterization of the prime numbers.

**Lemma 3.** Let \( p_1, \ldots, p_r \) be given distinct primes \( (r \geq 1) \). Then the equation

\[
T(k) = p_1 p_2 \cdots p_r
\]
is solvable if \( r = 1 \).

**Proof.** Since \( p_i | T(k) \), we get \( p_i | k \) for all \( i = 1, \ldots, r \). Thus \( p_1 \ldots p_r | k \), and Lemma 2 implies 

\[ T(p_1 \ldots p_r) | T(k) = p_1 \ldots p_r. \]

Since \( p_1 \ldots p_r | T(p_1 \ldots p_r) \), we have \( T(p_1 \ldots p_r) = p_1 \ldots p_r \), which by Theorem 8 is possible only if \( r = 1 \).

**Theorem 9.** Let \( P(n) \) denote the greatest prime factor of \( n > 1 \). If \( n \) is squarefree, then

\[ T_*(n) = P(n). \] (27)

**Proof.** Let \( n = p_1 p_2 \ldots p_r \), where \( p_1 < p_2 < \ldots < p_r \). If \( T(k) | (p_1 \ldots p_r) \), then clearly \( T(k) \in \{1, p_1, \ldots, p_r, p_1 p_2, \ldots, p_1 p_2 \ldots p_r\} \). By Lemma 3 we cannot have

\[ T(k) \in \{p_1 p_2, \ldots, p_1 p_2 \ldots p_r\}, \]

so \( T(k) \in \{1, p_1, \ldots, p_r\} \), when \( k \in \{1, p_1, \ldots, p_r\} \). The greatest \( k \) is \( p_r = P(n) \).

**Remark.** Therefore \( T_*(pq) = q \) for \( p < q \). For example, \( T_*(2 \cdot 7) = 7, T_*(3 \cdot 5) = 5 \), \( T_*(3 \cdot 7) = 7, T_*(2 \cdot 11) = 11 \), etc.

**Theorem 10.**

\[ T_*(p^\alpha) = p^\alpha (p = \text{prime}), \] (28)

where \( \alpha \) is the greatest integer with the property

\[ \frac{\alpha(\alpha + 1)}{2} \leq n. \] (29)

**Proof.** If \( T(k) | p^m \), then \( T(k) = p^m \) for \( m \leq n \). Let \( q \) be a prime divisor of \( k \). Then \( q = T(q) | T(k) = 2^m \) implies \( q = p \), so \( k = p^\alpha \). But then \( T(k) = p^{\alpha(\alpha + 1)/2} \) with \( \alpha \) the greatest number such that \( \alpha(\alpha + 1)/2 \leq n \), which finishes the proof of (28).

For example, \( T_*(4) = 2 \), since \( \frac{\alpha(\alpha + 1)}{2} \leq 2 \) gives \( \alpha_{\text{max}} = 1 \).

\[ T_*(16) = 4, \text{ since } \frac{\alpha(\alpha + 1)}{2} \leq 4 \text{ is satisfied with } \alpha_{\text{max}} = 2. \]

\[ T_*(9) = 3, \text{ and } T_*(27) = 9 \text{ since } \frac{\alpha(\alpha + 1)}{2} \leq 3 \text{ with } \alpha_{\text{max}} = 2. \]

**Theorem 11.** Let \( p, q \) be distinct primes. Then

\[ T_*(p^2 q) = \max\{p, q\}. \] (30)

**Proof.** If \( T(k) | p^2 q \), then \( T(k) \in \{1, p, q, p^2, pq, p^2 q\} \). The equations \( T(k) = p^2 \), \( T(k) = pq \), \( T(k) = p^2 q \) are impossible. For example, for the first equation, this can be proved as follows. By \( p | T(k) \) one has \( p | k \), so \( k = pm \). Then \( p(pm) \) are in \( T(k) \), so \( m = 1 \). But then \( T(k) = p \neq p^2 \).

For the last equation, \( k = (pq)m \) and \( pm(qm)(pqm) \) are in \( T(k) \), which is impossible.

**Theorem 12.** Let \( p, q \) be distinct primes. Then

\[ T_*(p^3 q) = \max\{p^2, q\}. \] (31)

**Proof.** As above, \( T(k) \in \{1, p, q, pq, p^2 q, p^3 q, p^2, p^3\} \) and \( T(k) \in \{pq, p^2 q, p^3 q, p^2, p^3\} \) are impossible. But \( T(k) = p^3 \) by Lemma 1 gives \( k^{d(k)} = p^3 \), so \( k = p^m \), when \( d(k) = m + 1 \). This gives \( m(m + 1) = 6 \), so \( m = 2 \). Thus \( k = p^2 \). Since \( p < p^2 \) the result follows.

**Remark.** The equation

\[ T(k) = p^s \] (32)
can be solved only if $k^{d(k)} = p^{2s}$, so $k = p^m$ and we get $m(m + 1) = 2s$. Therefore $k = p^m$, with $m(m + 1) = 2s$, if this is solvable. If $s$ is not a triangular number, this is impossible.

**Theorem 13.** Let $p, q$ be distinct primes. Then

$$\mathcal{T}_s(p^s q) = \begin{cases} \max\{p, q\}, & \text{if } s \text{ is not a triangular number}, \\ \max\{p^s, q\}, & \text{if } s = \frac{m(m+1)}{2}. \end{cases}$$

**References**


