A note on the Pseudo-Smarandache function

A.A.K. Majumdar
Department of Mathematics, Jahangirnagar University,
Savar, Dhaka 1342, Bangladesh

Abstract This paper gives some results and observations related to the Pseudo-Smarandache function $Z(n)$. Some explicit expressions of $Z(n)$ for some particular cases of $n$ are also given.

Keywords The Pseudo-Smarandache function, Smarandache perfect square, equivalent.

§1. Introduction

The Pseudo-Smarandache function $Z(n)$, introduced by Kashihara [1], is as follows:

Definition 1.1. For any integer $n \geq 1$, $Z(n)$ is the smallest positive integer $m$ such that $1 + 2 + \cdots + m$ is divisible by $n$. Thus,

$$Z(n) = \min \left\{ m : m \in \mathbb{N} : n \mid \frac{m(m+1)}{2} \right\}. \quad (1.1)$$

As has been pointed out by Ibstedt [2], an equivalent definition of $Z(n)$ is

Definition 1.2.

$$Z(n) = \min \left\{ k : k \in \mathbb{N} : \sqrt{1 + 8kn} \text{ is a perfect square} \right\}.$$

Kashihara [1] and Ibstedt [2] studied some of the properties satisfied by $Z(n)$. Their findings are summarized in the following lemmas:

Lemma 1.1. For any $m \in \mathbb{N}$, $Z(n) \geq 1$. Moreover, $Z(n) = 1$ if and only if $n = 1$, and $Z(n) = 2$ if and only if $n = 3$.

Lemma 1.2. For any prime $p \geq 3$, $Z(p) = p - 1$.

Lemma 1.3. For any prime $p \geq 3$ and any $k \in \mathbb{N}$, $Z(p^k) = p^k - 1$.

Lemma 1.4. For any $k \in \mathbb{N}$, $Z(2^k) = 2^{k+1} - 1$.

Lemma 1.5. For any composite number $n \geq 4$, $Z(n) \geq \max \{ Z(N) : N \mid n \}$.

In this paper, we give some results related to the Pseudo-Smarandache function $Z(n)$.

In §2, we present the main results of this paper. Simple explicit expressions for $Z(n)$ are available for particular cases of $n$. In Theorems 2.1 – 2.11, we give the expressions for $Z(2p)$, $Z(3p)$, $Z(2p^2)$, $Z(3p^2)$, $Z(2p^3)$, $Z(3p^3)$, $Z(4p)$, $Z(5p)$, $Z(6p)$, $Z(7p)$ and $Z(11p)$, where $p$ is a prime and $k \geq 3$ is an integer. Ibstedt [2] gives an expression for $Z(pq)$ where $p$ and $q$ are distinct primes. We give an alternative expressions for $Z(pq)$, which is more efficient from the computational point of view. This is given in Theorem 2.12, whose proof shows that the solution of $Z(pq)$ involves the solution of two Diophantine equations. Some particular cases of Theorem

---

1On Sabbatical leave from: Ritsumeikan Asia-Pacific University, 1-1 Jumonjibaru, Beppu-shi, Oita-ken, Japan.
2.12 are given in Corollaries 2.1 - 2.16. We conclude this paper with some observations about the properties of $Z(n)$, given in four Remarks in the last §3.

§2. Main Results

We first state and prove the following results.

**Lemma 2.1.** Let $n = \frac{k(k+1)}{2}$ for some $k \in \mathbb{N}$. Then, $Z(n) = k$.

**Proof.** Noting that $k(k+1) = m(m+1)$ if and only if $k = m$, the result follows. The following lemma gives lower and upper bounds of $Z(n)$.

**Lemma 2.2.** $3 \leq n \leq 2n-1$ for all $n \geq 4$.

**Proof.** Letting $f(m) = \frac{m(m+1)}{2}, m \in \mathbb{N}$, see that $f(m)$ is strictly increasing in $m$ with $f(2) = 3$. Thus, $Z(n) = 2$ if and only if $n = 3$. This, together with Lemma 1.1, gives the lower bound of $Z(n)$ for $n \geq 4$. Again, since $n \mid f(2n-1)$, it follows that $Z(n)$ cannot be greater than $2n-1$. Since $Z(n) = 2n-1$ if $n = 2k$ for some $k \in \mathbb{N}$, it follows that the upper bound of $Z(n)$ in Lemma 2.2 cannot be improved further. However, the lower bound of $Z(n)$ can be improved. For example, since $f(4) = 10$, it follows that $Z(n) \geq 5$ for all $n \geq 11$. A better lower bound of $Z(n)$ is given in Lemma 1.5 for the case when $n$ is a composite number. In Theorems 2.1 - 2.4, we give expressions for $Z(2p), Z(3p), Z(2p^2)$ and $Z(3p^2)$ where $p \geq 5$ is a prime. To prove the theorems, we need the following results.

**Lemma 2.3.** Let $p$ be a prime. Let an integer $n(\geq p)$ be divisible by $p^k$ for some integer $k(\geq 1)$. Then, $p^k$ does not divide $n+1$ (and $n-1$).

**Lemma 2.4.** $6 \mid n(n+1)(n+2)$ for any $n \in \mathbb{N}$. In particular, $6 \mid (p^2 - 1)$ for any prime $p \geq 5$.

**Proof.** The first part is a well-known result. In particular, for any prime $p \geq 5$, $6 \mid (p-1)(p+1)$. But since $p(\geq 5)$ is not divisible by 6, it follows that $6 \mid (p-1)(p+1)$.

**Theorem 2.1.** If $p \geq 5$ is a prime, then

$$Z(2p) = \begin{cases} p - 1, & \text{if } 4 \mid (p-1); \\ p, & \text{if } 4 \mid (p+1). \end{cases}$$

**Proof.**

$$Z(2p) = \min \left\{ m : 2p \mid \frac{m(m+1)}{2} \right\} = \min \left\{ m : p \mid \frac{m(m+1)}{4} \right\}. \quad (1)$$

If $p \mid m(m+1)$, then $p$ must divide either $m$ or $m+1$, but not both (by Lemma 2.3). Thus, the minimum $m$ in (1) may be taken as $p - 1$ or $p$ depending on whether $p - 1$ or $p + 1$ respectively is divisible by 4. We now consider the following two cases that may arise :

Case 1 : $p$ is of the form $p=4a+1$ for some integer $a \geq 1$. In this case, $4 \mid (p-1)$, and hence, $Z(2p) = p - 1$.

Case 2 : $p$ is of the form $p = 4a + 3$ for some integer $a \geq 1$. Here, $4 \mid (p+1)$ and hence, $Z(2p) = p$. 


Theorem 2.2. If $p \geq 5$ is a prime, then

$$Z(3p) = \begin{cases} p - 1, & \text{if } 3 \mid (p - 1); \\ p, & \text{if } 3 \mid (p + 1). \end{cases} \quad (1)$$

Proof.

$$Z(3p) = \min \left\{ m : 3p \mid \frac{m(m + 1)}{2} \right\} = \min \left\{ m : p \mid \frac{m(m + 1)}{6} \right\}. \quad (2)$$

If $p \mid m(m + 1)$, then $p$ must divide either $m$ or $m + 1$, but not both (by Lemma 2.3). Thus, the minimum $m$ in (2) may be taken as $p - 1$ or $p$ according as $p - 1$ or $p + 1$ respectively is divisible by 6. But, since both $p - 1$ and $p + 1$ are divisible by 2, it follows that the minimum $m$ in (2) may be taken as $p - 1$ or $p$ according as $p - 1$ or $p + 1$ respectively is divisible by 3.

We now consider the following two possible cases that may arise:

Case 1 : $p$ is of the form $p = 3a + 1$ for some integer $a \geq 1$. In this case, $3 \mid (p - 1)$, and hence, $Z(3p) = p - 1$.

Case 2 : $p$ is of the form $p = 3a + 2$ for some integer $a \geq 1$. Here, $3 \mid (p + 1)$, and hence, $Z(3p) = p$.

Theorem 2.3. If $p \geq 3$ is a prime, then $Z(2p^2) = p^2 - 1$.

Proof.

$$Z(2p^2) = \min \left\{ m : 2p^2 \mid \frac{m(m + 1)}{2} \right\} = \min \left\{ m : p^2 \mid \frac{m(m + 1)}{4} \right\}. \quad (3)$$

If $p^2 \mid m(m + 1)$, then $p^2$ must divide either $m$ or $m + 1$, but not both (by Lemma 2.3). Thus, the minimum $m$ in (3) may be taken as $p^2 - 1$ if $p^2 - 1$ is divisible by 4. But, since both $p - 1$ and $p + 1$ are divisible by 2, it follows that $4 \mid (p - 1)(p + 1)$. Hence, $Z(2p^2) = p^2 - 1$.

Theorem 2.4. If $p \geq 5$ is a prime, then $Z(3p^2) = p^2 - 1$.

Proof.

$$Z(3p^2) = \min \left\{ m : 3p^2 \mid \frac{m(m + 1)}{2} \right\} = \min \left\{ m : p^2 \mid \frac{m(m + 1)}{6} \right\}. \quad (4)$$

If $p^2 \mid m(m + 1)$, then $p^2$ must divide either $m$ or $m + 1$, but not both (by Lemma 2.3). Thus, the minimum $m$ in (4) may be taken as $p^2 - 1$ if $p^2 - 1$ is divisible by 6. By Lemma 2.4, $6 \mid (p^2 - 1)$. Consequently, $Z(2p^2) = p^2 - 1$.

Definition 2.1. A function $g : N \to N$ is called multiplicative if and only if $g(n_1n_2) = g(n_1)g(n_2)$ for all $n_1, n_2 \in N$ with $(n_1, n_2) = 1$.

Remark 2.1. From Lemma 2.2 and Theorem 2.1, we see that $Z(2p) \neq 3(p - 1) = Z(2)Z(p)$ for any odd prime $p$. Moreover, $Z(3p^2) = p^2 - 1 \neq Z(2p^2) + Z(p^2)$. These show that $Z(n)$ is neither additive nor multiplicative, as has already been noted by Kasihara [1]. The expressions for $Z(2p^k)$ and $Z(3p^k)$ for $k \geq 3$ are given in Theorem 2.5 and Theorem 2.6 respectively. For the proofs, we need the following results:

Lemma 2.5.

(1) $4$ divides $3^2k - 1$ for any integer $k \geq 1$.

(2) $4$ divides $3^{2k+1} + 1$ for any integer $k \geq 0$. 
Proof.
(1) Writing $3^{2k} - 1 = (3k - 1)(3k + 1)$, the result follows immediately.
(2) The proof is by induction on $k$. The result is clearly true for $k = 0$. So, we assume that the result is true for some integer $k$, so that 4 divides $3^{2k+1} + 1$ for some $k$. Now, since $3^{2k+3} + 1 = 9(3^{2k+1} + 1) - 8$, it follows that 4 divides $3^{2k+3} + 1$, completing the induction.

**Lemma 2.6.**
(1) 3 divides $2^k - 1$ for any integer $k \geq 1$.
(2) 3 divides $2^k + 1$ for any integer $k \geq 0$.

**Proof.**
(1) By Lemma 2.4, 3 divides $(2k - 1)(2k + 1)$. Since 3 does not divide 2, 3 must divide $(2k - 1)(2k + 1) = 2^k k - 1$.
(2) The result is clearly true for $k = 0$. To prove by induction, the induction hypothesis is that 3 divides $2^k + 1$ for some $k$. Now, since $2^{k+3} + 1 = 4(3^{k+1} + 1) - 3$, it follows that 3 divides $2^{k+3} + 1$, so that the result is true for $k + 1$ as well, completing the induction.

**Theorem 2.5.** If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$Z(2p^k) = \begin{cases} p^k, & \text{if } 4 \mid (p - 1) \text{ and } k \text{ is odd;} \\ p^k - 1, & \text{otherwise.} \end{cases}$$

**Proof.**

$$Z(2p^k) = \min \left\{ m : 2p^k \mid \frac{m(m + 1)}{2} \right\} = \min \left\{ m : p^k \mid \frac{m(m + 1)}{4} \right\}. \quad (5)$$

If $p^k \mid m(m+1)$, then $p^k$ must divide either $m$ or $m+1$, but not both (by Lemma 2.3). Thus, the minimum $m$ in (5) may be taken as $p^k - 1$ or $p^k$ according as $p^k - 1$ or $p^k$ is respectively divisible by 4. We now consider the following two possibilities:

Case 1: $p$ is of the form $4a + 1$ for some integer $a \geq 1$. In this case, $p^k = (4a + 1)^k = (4a)^k + C_k^1(4a)^{k-1} + \cdots + C_k^{k-1}(4a) + 1$, showing that $4 \mid (p^k - 1)$. Hence, in this case, $Z(2p^k) = p^k - 1$.

Case 2: $p$ is of the form $4a + 3$ for some integer $a \geq 1$. Here, $p^k = (4a + 3)^k = (4a)^k + C_k^1(4a)^{k-1}3 + \cdots + C_k^{k-1}(4a)3^{k-1} + 3^k$.

(1) If $k \geq 2$ is even, then by Lemma 2.5, $4 \mid (3^k - 1)$, so that $4 \mid (p^k - 1)$. Thus, $Z(2p^k) = p^k - 1$.

(2) If $k \geq 3$ is odd, then by Lemma 2.5, $4 \mid (3^k + 1)$, and so $4 \mid (p^k + 1)$. Hence, $Z(2p^k) = p^k$.

All these complete the proof of the theorem.

**Theorem 2.6.** If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$Z(3p^k) = \begin{cases} p^k, & \text{if } 3 \mid (p + 1) \text{ and } k \text{ is odd;} \\ p^k - 1, & \text{otherwise.} \end{cases}$$

**Proof.**

$$Z(3p^k) = \min \left\{ m : 3p^k \mid \frac{m(m + 1)}{2} \right\} = \min \left\{ m : p^k \mid \frac{m(m + 1)}{6} \right\}. \quad (6)$$
If \( p^k \mid m(m+1) \), then \( p^k \) must divide either \( m \) or \( m+1 \), but not both (by Lemma 2.3). Thus, the minimum \( m \) in (6) may be taken as \( p^k - 1 \) or \( p^k \) according as \( p^k - 1 \) or \( p^k \) is respectively divisible by 6. We now consider the following two possible cases:

Case 1: \( p \) is of the form \( 3a + 1 \) for some integer \( a \geq 1 \). In this case, \( p^k = (3a + 1)^k = (3a)^k + C_k^1(3a)^{k-1} + \cdots + C_k^{k-1}(3a) + 1 \), it follows that \( 3 \mid (p^k - 1) \). Thus, in this case, \( Z(3p^k) = p^k - 1 \).

Case 2: \( p \) is of the form \( 3a + 2 \) for some integer \( a \geq 1 \). Here, \( p^k = (3a + 2)^k = (3a)^k + C_k^1(3a)^{k-1}(2) + \cdots + C_k^{k-1}(3a)2^{k-1} + 2^k \).

(1) If \( k \geq 2 \) is even, then by Lemma 2.6, \( 3 \mid (2^k - 1) \), so that \( 3 \mid (p^k - 1) \). Thus, \( Z(3p^k) = p^k - 1 \).

(2) If \( k \geq 3 \) is odd, then by Lemma 2.6, \( 3 \mid (2^k + 1) \), and so \( 3 \mid (p^k + 1) \). Thus, \( Z(3p^k) = p^k \).

In Theorem 2.7 - Theorem 2.9, we give the expressions for \( Z(4p) \), \( Z(5p) \) and \( Z(6p) \) respectively, where \( p \) is a prime. Note that, each case involves 4 possibilities.

**Theorem 2.7.** If \( p \geq 5 \) is a prime, then

\[
Z(4p) = \begin{cases} 
  p - 1, & \text{if } 8 \mid (p - 1); \\
  p, & \text{if } 8 \mid (p + 1); \\
  3p - 1, & \text{if } 8 \mid (3p - 1); \\
  3p, & \text{if } 8 \mid (3p + 1).
\end{cases}
\]

Proof.

\[
Z(4p) = \min \left\{ m : 4p \mid \frac{m(m+1)}{2} \right\} = \min \left\{ m : p \mid \frac{m(m+1)}{8} \right\}.
\] (7)

If \( p \mid m(m+1) \), then \( p \) must divide either \( m \) or \( m+1 \), but not both (by Lemma 2.3), and then \( 8 \) must divide either \( p-1 \) or \( p+1 \). In the particular case when \( 8 \) divides \( p-1 \) or \( p+1 \), the minimum \( m \) in (7) may be taken as \( p-1 \) or \( p+1 \) respectively. We now consider the following four cases may arise:

Case 1: \( p \) is of the form \( p = 8a + 1 \) for some integer \( a \geq 1 \). In this case, \( 8 \mid (p - 1) \), and hence \( Z(4p) = p - 1 \).

Case 2: \( p \) is of the form \( p = 8a + 7 \) for some integer \( a \geq 1 \). Here, \( 8 \mid (p + 1) \), and hence \( Z(4p) = p \).

Case 3: \( p \) is of the form \( p = 8a + 3 \) for some integer \( a \geq 1 \). In this case, \( 8 \mid (3p - 1) \), and hence \( Z(4p) = 3p - 1 \).

Case 4: \( p \) is of the form \( p = 8a + 5 \) for some integer \( a \geq 1 \). Here, \( 8 \mid (3p + 1) \), and hence \( Z(4p) = 3p \).

**Theorem 2.8.** If \( p \geq 7 \) is a prime, then

\[
Z(5p) = \begin{cases} 
  p - 1, & \text{if } 10 \mid (p - 1); \\
  p, & \text{if } 10 \mid (p + 1); \\
  2p - 1, & \text{if } 5 \mid (2p - 1); \\
  2p, & \text{if } 5 \mid (2p + 1).
\end{cases}
\]
Proof.

\[ Z(5p) = \min \left\{ m : 5p \mid \frac{m(m+1)}{2} \right\} = \min \left\{ m : p \mid \frac{m(m+1)}{10} \right\}. \] (8)

If \( p \mid m(m+1) \), then \( p \) must divide either \( m \) or \( m+1 \), but not both (by Lemma 2.3), and then 5 must divide either \( m-1 \) or \( m+1 \). In the particular case when 5 divides \( p-1 \) or \( p+1 \), the minimum \( m \) in (8) may be taken as \( p-1 \) or \( p+1 \) respectively. We now consider the four cases below that may arise:

Case 1: \( p \) is a prime whose last digit is 1. In this case, 10 \( \mid (p-1) \), and hence \( Z(5p) = p-1 \).

Case 2: \( p \) is a prime whose last digit is 9. In such a case, 10 \( \mid (p+1) \), and so \( Z(5p) = p \).

Case 3: \( p \) is a prime whose last digit is 3. In this case, 5 \( \mid (2p-1) \). Thus, the minimum \( m \) in (9) may be taken as \( 2p-1 \). Hence \( Z(5p) = 2p-1 \).

Case 4: \( p \) is a prime whose last digit is 7. Here, 5 \( \mid (2p+1) \), and hence \( Z(5p) = 2p \).

Theorem 2.9. If \( p \geq 5 \) is a prime, then

\[ Z(6p) = \begin{cases} p-1, & \text{if } 12 \mid (p-1); \\ p, & \text{if } 12 \mid (p+1); \\ 2p-1, & \text{if } 4 \mid (3p+1); \\ 2p, & \text{if } 4 \mid (3p-1). \end{cases} \]

Proof.

\[ Z(6p) = \min \left\{ m : 6p \mid \frac{m(m+1)}{2} \right\} = \min \left\{ m : p \mid \frac{m(m+1)}{12} \right\}. \] (9)

If \( p \mid m(m+1) \), then \( p \) must divide either \( m \) or \( m+1 \), but not both (by Lemma 2.3), and then 12 must divide either \( m-1 \) or \( m+1 \). In the particular case when 12 divides \( p-1 \) or \( p+1 \), the minimum \( m \) in (9) may be taken as \( p-1 \) or \( p+1 \) respectively. We now consider the four cases that may arise:

Case 1: \( p \) is of the form \( p = 12a+1 \) for some integer \( a \geq 1 \). In this case, 12 \( \mid (p-1) \), and hence \( Z(6p) = p-1 \).

Case 2: \( p \) is of the form \( p = 12a+11 \) for some integer \( a \geq 1 \). Here, 12 \( \mid (p+1) \), and hence \( Z(6p) = p \).

Case 3: \( p \) is of the form \( p = 12a+5 \) for some integer \( a \geq 1 \). In this case, 4 \( \mid (3p+1) \). The minimum \( m \) in (10) may be taken as 3p, and hence \( Z(6p) = 3p \).

Case 4: \( p \) is of the form \( p = 12a+7 \) for some integer \( a \geq 1 \). Here, 4 \( \mid (3p-1) \), and hence \( Z(6p) = 3p-1 \).

It is possible to find explicit expressions for \( Z(7p) \) or \( Z(11p) \), where \( p \) is a prime, as are given in Theorem 2.10 and Theorem 2.11 respectively, but it becomes more complicated. For example, in finding the expression for \( Z(7p) \), we have to consider all the six possibilities, while the expression for \( Z(11p) \) involves 10 alternatives.
Theorem 2.10. If \( p \geq 11 \) is a prime, then

\[
Z(7p) = \begin{cases} 
  p - 1, & \text{if } 7 \mid (p - 1); \\
  p, & \text{if } 7 \mid (p + 1); \\
  2p - 1, & \text{if } 7 \mid (2p - 1); \\
  2p, & \text{if } 7 \mid (2p + 1); \\
  3p - 1, & \text{if } 7 \mid (3p - 1); \\
  3p, & \text{if } 7 \mid (3p + 1).
\end{cases}
\]

Proof:

\[
Z(7p) = \min\{m : 7p \frac{m(m + 1)}{2} \} = \min\{m : p \frac{m(m + 1)}{14} \}.
\] (10)

If \( p|m(m + 1) \), then \( p \) must divide either \( m \) or \( m + 1 \), but not both (by Lemma 2.3), and then 7 must divide either \( m + 1 \) or \( m \) respectively. In the particular case when 12 divides \( p - 1 \) or \( p + 1 \), the minimum \( m \) in (10) may be taken as \( p - 1 \) or \( p \) respectively. We now consider the following six cases that may arise:

Case 1 : \( p \) is of the form \( p = 7a + 1 \) for some integer \( a \geq 1 \). In this case, \( 7|(p - 1) \). Therefore, \( Z(7p) = p - 1 \).

Case 2 : \( p \) is of the form \( p = 7a + 6 \) for some integer \( a \geq 1 \). Here, \( 7|(p + 1) \), and so, \( Z(7p) = p \).

Case 3 : \( p \) is of the form \( p = 7a + 2 \) for some integer \( a \geq 1 \), so that \( 7|(3p + 1) \). In this case, the minimum \( m \) in (11) may be taken as \( 3p \). That is, \( Z(7p) = 3p \).

Case 4 : \( p \) is of the form \( p = 7a + 5 \) for some integer \( a \geq 1 \). Here, \( 7|(3p - 1) \), and hence, \( Z(7p) = 3p - 1 \).

Case 5 : \( p \) is of the form \( p = 7a + 3 \) for some integer \( a \geq 1 \). In this case, \( 7|(2p + 1) \), and hence, \( Z(7p) = 2p \).

Case 6 : \( p \) is of the form \( p = 7a + 4 \) for some integer \( a \geq 1 \). Here, \( 7|(2p - 1) \), and hence, \( Z(7p) = 2p - 1 \).

Theorem 2.11. For any prime \( p \geq 13 \),

\[
Z(7p) = \begin{cases} 
  p - 1, & \text{if } 11 \mid (p - 1); \\
  p, & \text{if } 11 \mid (p + 1); \\
  2p - 1, & \text{if } 11 \mid (2p - 1); \\
  2p, & \text{if } 11 \mid (2p + 1); \\
  3p - 1, & \text{if } 11 \mid (3p - 1); \\
  3p, & \text{if } 11 \mid (3p + 1); \\
  4p - 1, & \text{if } 11 \mid (4p - 1); \\
  4p, & \text{if } 11 \mid (4p + 1); \\
  5p - 1, & \text{if } 11 \mid (5p - 1); \\
  5p, & \text{if } 11 \mid (5p + 1).
\end{cases}
\]
Proof:

$$Z(11p) = \min\{m : 11p \frac{m(m+1)}{2} \} = \min\{m : p \frac{m(m+1)}{22} \}.$$  \hfill (11)

If $p|m(m+1)$, then $p$ must divide either $m$ or $m+1$, but not both (by Lemma 2.3), and then $11$ must divide either $m+1$ or $m$ respectively. In the particular case when $11$ divides $p-1$ or $p+1$, the minimum $m$ in (11) may be taken as $p-1$ or $p$ respectively. We have to consider the ten possible cases that may arise:

Case 1 : $p$ is of the form $p = 11a + 1$ for some integer $a \geq 1$. In this case, $11|(p-1)$, and so, $Z(11p) = p-1$.

Case 2 : $p$ is of the form $p = 11a + 10$ for some integer $a \geq 1$. Here, $11|(p+1)$, and hence, $Z(11p) = p$.

Case 3 : $p$ is of the form $p = 11a + 2$ for some integer $a \geq 1$. In this case, $11|(5p+1)$, and hence, $Z(11p) = 5p$.

Case 4 : $p$ is of the form $p = 11a + 9$ for some integer $a \geq 1$. Here, $11|(5p-1)$, and hence, $Z(11p) = 5p-1$.

Case 5 : $p$ is of the form $p = 11a + 3$ for some integer $a \geq 1$. In this case, $11|(4p-1)$, and hence, $Z(11p) = 4p-1$.

Case 6 : $p$ is of the form $p = 11a + 8$ for some integer $a \geq 1$. Here, $11|(4p+1)$, and hence, $Z(11p) = 4p$.

Case 7 : $p$ is of the form $p = 11a + 4$ for some integer $a \geq 1$. In this case, $11|(3p-1)$, and hence, $Z(11p) = 3p-1$.

Case 8 : $p$ is of the form $p = 11a + 7$ for some integer $a \geq 1$. Here, $11|(3p+1)$, and hence, $Z(11p) = 3p$.

Case 9 : $p$ is of the form $p = 11a + 5$ for some integer $a \geq 1$. In this case, $11|(2p+1)$, and hence, $Z(11p) = 2p$.

Case 10 : $p$ is of the form $p = 11a + 6$ for some integer $a \geq 1$. Here, $11|(2p-1)$, and hence, $Z(11p) = 2p-1$.

In Theorem 2.12, we give an expression for $Z(pq)$, where $p$ and $q$ are two distinct primes. In this connection, we state the following lemma. The proof of the lemma is similar to, for example, Theorem 12.2 of Gioia [3], and is omitted here.

**Lemma 2.7.** Let $p$ and $q$ be two distinct primes. Then, the Diophantine equation

$$qy - px = 1$$

has an infinite number of solutions. Moreover, if $(x_0, y_0)$ is a solution of the Diophantine equation, then any solution is of the form

$$x = x_0 + qt, y = y_0 + pt,$$

where $t \geq 0$ is an integer.

**Theorem 2.12.** Let $p$ and $q$ be two primes with $q > p \geq 5$. Then,

$$Z(pq) = \min\{qy_0 - 1, px_0 - 1\},$$
where 
\[ y_0 = \min\{y : x, y \in \mathbb{N}, qy - px = 1\}, \]
\[ x_0 = \min\{x : x, y \in \mathbb{N}, px - qy = 1\}. \]

**Proof:** Since 
\[ Z(pq) = \min\{m : pq|m(m+1)/2\}, \tag{12} \]
it follows that we have to consider the three cases below that may arise:

**Case 1:** When \( p | m \) and \( q | (m + 1) \). In this case, \( m = px \) for some integer \( x \geq 1 \), \( m + 1 = qy \) for some integer \( y \geq 1 \). From these two equations, we get the Diophantine equation
\[ qy - px = 1. \]
By Lemma 2.7, the above Diophantine equation has infinite number of solutions. Let
\[ y_0 = \min\{y : x, y \in \mathbb{N}, qy - px = 1\}. \]
For this \( y_0 \), the corresponding \( x_0 \) is given by the equation \( qy_0 - p_0x = 1 \). Note that \( y_0 \) and \( x_0 \) cannot be both odd or both even. Then, the minimum \( m \) in (12) is given by
\[ m + 1 = qy_0 \Rightarrow m = qy_0 - 1. \]

**Case 2:** When \( p | (m + 1) \) and \( q | m \). Here, \( m + 1 = px \) for some integer \( x \geq 1 \), \( m = qy \) for some integer \( y \geq 1 \). These two equations lead to the Diophantine equation. \( px - qy = 1 \). Let
\[ x_0 = \min\{x : x, y \in \mathbb{N}, px - qy = 1\}. \]
For this \( x_0 \), the corresponding \( y_0 \) is given by \( y_0 = (px_0 - 1)/q \). Here also, \( x_0 \) and \( y_0 \) both cannot be odd or even simultaneously. The minimum \( m \) in (12) is given by
\[ m + 1 = px_0 \Rightarrow m = px_0 - 1. \]

**Case 3:** When \( pq | (m + 1) \). In this case, \( m = pq - 1 \). But then, by Case 1 and Case 2 above, this does not give the minimum \( m \). Thus, this case cannot occur. The proof of the theorem now follows by virtue of Case 1 and Case 2.

**Remark 2.2.** Let \( p \) and \( q \) be two primes with \( q \geq p \geq 5 \). Let \( q = kp + \ell \) for some integers \( k \) and \( \ell \) with \( k \geq 1 \) and \( 1 \leq \ell \leq p-1 \). We now consider the two cases given in Theorem 2.12:

**Case 1:** When \( p | m \) and \( q | (m + 1) \). In this case, \( m = px \) for some integer \( x \geq 1 \), \( m + 1 = qy = (kp + \ell)y \) for some integer \( y \geq 1 \). From these two equations, we get
\[ \ell y - (x - ky)p = 1 \tag{2.1} \]

**Case 2:** When \( p | (m + 1) \) and \( q | m \). Here, \( m + 1 = px \) for some integer \( x \geq 1 \), \( m = (kp + \ell)y \) for some integer \( y \geq 1 \). These two equations lead to
\[ (x - ky)p - \ell y = 1 \tag{2.2}. \]
In some particular cases, explicit expressions of \( Z(pq) \) may be found. These are given in the following corollaries.

**Corollary 2.1.** Let \( p \) and \( q \) be two primes with \( q > p \geq 5 \). Let \( q = kp + 1 \) for some integer \( k \geq 2 \). Then, \( Z(pq) = q - 1 \).

**Proof.** From (2.1) with \( \ell = 1 \), we get \( y - (x - ky)p = 1 \), the minimum solution of which is \( y = 1 \), \( x = ky = k \). Then, the minimum \( m \) in (12) is given by

\[
m + 1 = qy = q \Rightarrow m = q - 1.
\]

Note that, from (2.2) with \( \ell = 1 \), we have \( (x - ky)p - y = 1 \), with the least possible solution \( y = p - 1 \) (and \( x = ky = 1 \)).

**Corollary 2.2.** Let \( p \) and \( q \) be two primes with \( q > p \geq 5 \). Let \( q = (k + 1)p - 1 \) for some integer \( k \geq 1 \).

Then, \( Z(pq) = q \).

**Proof.** From (2.2) with \( \ell = p - 1 \), we have, \( y - [(k + 1)y - x]p = 1 \), the minimum solution of which is \( y = 1 \), \( x = (k + 1)y = k + 1 \). Then, the minimum \( m \) in (12) is given by \( m = qy = q \).

Note that, from (2.1) with \( \ell = p - 1 \), we have \( [(k + 1)y - x]p - x = 1 \) with the least possible solution \( y = p - 1 \) (and \( (k + 1)y - x = 1 \)).

**Corollary 2.3.** Let \( p \) and \( q \) be two primes with \( q > p \geq 5 \). Let \( q = kp + 2 \) for some integer \( k \geq 1 \). Then,

\[
Z(pq) = \frac{q(p - 1)}{2}.
\]

**Proof.** From (2.2) with \( \ell = 2 \), we have \( (x - ky)p - 2y = 1 \), with the minimum solution \( y = \frac{k+1}{2} \) (and \( x = ky+1 \)). This gives \( m = qy = 1 = \frac{q(p+1)}{2} - 1 \) as another possible solution of (12). Now, since \( \frac{q(p+1)}{2} - 1 > \frac{q(p-1)}{2} \), it follows that

\[
Z(pq) = \frac{q(p - 1)}{2},
\]

which we intended to prove.

**Corollary 2.4.** Let \( p \) and \( q \) be two primes with \( q > p \geq 5 \). Let \( q = (k + 1)p - 2 \) for some integer \( k \geq 1 \). Then,

\[
Z(pq) = \frac{q(p - 1)}{2} - 1.
\]

**Proof.** By (2.1) with \( \ell = p - 2 \), we get \( [(k + 1)y - x]p - 2y = 1 \), whose minimum solution is \( y = \frac{k+1}{2} \) (and \( x = (k+1)y = 1 \)). This gives \( m = qy - 1 = \frac{q(p-1)}{2} - 1 \) as one possible solution of (12). Note that, (2.2) with \( \ell = p - 2 \) gives \( 2y - [(k + 1)y - x]p = 1 \), with the minimum solution \( y = \frac{k+1}{2} \) (and \( x = (k + 1)y = 1 \)). Corresponding to this case, we get \( m = qy = \frac{q(p+1)}{2} \) as another possible solution of (12). But since \( \frac{q(p+1)}{2} > \frac{q(p-1)}{2} - 1 \), it follows that \( Z(pq) = \frac{q(p-1)}{2} - 1 \), establishing the theorem.

**Corollary 2.5.** Let \( p \) and \( q \) be two primes with \( q > p \geq 7 \). Let \( q = kp + 3 \) for some integer \( k \geq 1 \). Then,

\[
Z(pq) = \begin{cases} \frac{q(p-1)}{3}, & \text{if } 3 || (p-1); \\ \frac{q(p+1)}{3} - 1, & \text{if } 3 || (p+1). \end{cases}
\]
Proof. From (2.1) and (2.2) with $\ell = 3$, we have respectively
\begin{align*}
3y - (x - ky)p &= 1, \quad \text{(13)} \\
(x - ky)p - 3y &= 1. \quad \text{(14)}
\end{align*}
We now consider the following two possible cases:

Case 1 : When 3 divides $p - 1$.
In this case, the minimum solution is obtained from (14), which is $y = \frac{p-1}{3}$ (and $x - ky = 1$).
Also, $p - 1$ is divisible by 2 as well. Therefore, the minimum $m$ in (12) is $m = qy = \frac{q(p-1)}{3}$.

Case 2 : When 3 divides $p + 1$.
In this case, (13) gives the minimum solution, which is $y = \frac{p+1}{3}$ (and $x - ky = 1$). Note that, 2 divides $p + 1$. Therefore, the minimum $m$ in (12) is $m = qy - 1 = \frac{q(p+1)}{3} - 1$.
Thus, the theorem is established.

Corollary 2.6. Let $p$ and $q$ be two primes with $q > p \geq 7$. Let $q = (k + 1)p - 3$ for some integer $k \geq 1$. Then,
\begin{align*}
\left(\frac{p+1}{3}\right) - 1, &\text{ if } 3|(p+1); \\
\left(\frac{p-1}{3}\right) - 1, &\text{ if } 3|(p-1).
\end{align*}

Proof. From (2.1) and (2.2) with $\ell = p - 3$, we have respectively
\begin{align*}
[(k + 1)y - x)p - 3y &= 1, \quad \text{(15)} \\
3y - [(k + 1)y - x]p &= 1. \quad \text{(16)}
\end{align*}
We now consider the following two cases:

Case 1 : When 3 divides $p + 1$.
In this case, the minimum solution, obtained from (14), is $y = \frac{p+1}{3}$ (and $x = (k + 1)y = 1$)
Moreover, 2 divides $p + 1$. Therefore, the minimum $m$ in (12) is $m = qy = \frac{q(p+1)}{3}$.

Case 2 : When 3 divides $p - 1$.
In this case, the minimum solution, obtained from (13), is $y = \frac{p-1}{3}$ (and $x = (k + 1)y = 1$)
Moreover, 2 divides $p - 1$. Therefore, the minimum $m$ in (12) is $m = qy = \frac{q(p-1)}{3} - 1$.

Corollary 2.7. Let $p$ and $q$ be two primes with $q > p \geq 7$. Let $q = kp + 4$ for some integer $k \geq 1$. Then,
\begin{align*}
\frac{q(p+1)}{4} - 1, &\text{ if } 4|(p+1); \\
\frac{q(p-1)}{4} - 1, &\text{ if } 4|(p-1).
\end{align*}

Proof. From (2.1) and (2.2) with $\ell = 4$, we have respectively
\begin{align*}
4y - (x - ky)p &= 1, \quad \text{(17)} \\
(x - ky)p - 4y &= 1. \quad \text{(18)}
\end{align*}
Now, for any prime $p \geq 7$, exactly one of the following two cases can occur : Either $p - 1$ is divisible by 4, or $p + 1$ is divisible by 4. We thus consider the two possibilities separately below:

Case 1 : When 4 divides $p - 1$. 

In this case, the minimum solution is obtained from (18), is \( y = \frac{p-1}{4} \) (and \( x = ky + 1 \)). Therefore, the minimum \( m \) in (12) is is \( m = qy = \frac{q(p+1)}{4} \).

Case 2: When 4 divides \( p + 1 \).

In this case, (17) gives the minimum solution, which is \( y = \frac{p+1}{4} \) (and \( x = ky + 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{q(p+1)}{4} - 1 \).

**Corollary 2.8.** Let \( p \) and \( q \) be two primes with \( q > p \geq 7 \). Let \( q = (k+1)p - 4 \) for some integer \( k \geq 1 \). Then,

\[
Z(pq) = \begin{cases} 
\frac{q(p+1)}{4}, & \text{if } 4|(p+1); \\
\frac{q(p-1)}{4} - 1, & \text{if } 4|(p-1).
\end{cases}
\]

**Proof.** From (2.1) and (2.2) with \( \ell = p - 4 \), we have respectively

\[
[(k+1)y - x]p - 4y = 1, \quad (19)
\]

\[
4y - [(k+1)y - x]p = 1. \quad (20)
\]

We now consider the following two cases which are the only possibilities (as noted in the proof of Corollary 2.7).

Case 1: When 4 divides \( p - 1 \).

In this case, the minimum solution obtained from (20) is \( y = \frac{p-1}{4} \) (and \( x = (k+1)y - 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy = \frac{q(p+1)}{4} \).

Case 2: When 4 divides \( p - 1 \).

In this case, the minimum solution, obtained from (19), is \( y = \frac{p-1}{4} \) (and \( x = (k+1)y - 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy = \frac{q(p-1)}{4} - 1 \).

**Corollary 2.9.** Let \( p \) and \( q \) be two primes with \( q > p \geq 11 \). Let \( q = kp + 5 \) for some integer \( k \geq 1 \). Then,

\[
Z(pq) = \begin{cases} 
\frac{q(p-1)}{5}, & \text{if } 5|(p-1); \\
q(2a + 1) - 1, & \text{if } p = 5a + 2; \\
q(2a + 1), & \text{if } p = 5a + 3; \\
\frac{q(p+1)}{5} - 1, & \text{if } 5|(p+1).
\end{cases}
\]

**Proof.** From (2.1) and (2.2) with \( \ell = 5 \), we have respectively

\[
5y - (x - ky)p = 1, \quad (21)
\]

\[
(x - ky)p - 5y = 1. \quad (22)
\]

Now, for any prime \( p \geq 7 \), exactly one of the following four cases occur:

Case 1: When \( p \) is of the form \( p = 5a + 1 \) for some integer \( a \geq 2 \).

In this case, 5 divides \( p - 1 \). Then, the minimum solution is obtained from (22) which is \( y = \frac{p-1}{5} \) (and \( x - ky = 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy = \frac{q(p-1)}{5} \).

Case 2: When \( p \) is of the form \( p = 5a + 2 \) for some integer \( a \geq 2 \).

In this case, from (21) and (22), we get respectively

\[
1 = 5y - (x - ky)(5a + 2) = 5[y - (x - ky)a] - 2(x - ky), \quad (23)
\]
Clearly, the minimum solution is obtained from (23), which is
\[ y - (x - ky)a = 1, \quad x - ky = 2 \implies y = 2a + 1 \quad \text{and} \quad x = k(2a + 1) + 2. \]
Hence, in this case, the minimum \( m \) in (12) is \( m = qy - 1 = q(2a + 1) - 1 \).

Case 3: When \( p \) is of the form \( p = 5a + 3 \) for some integer \( a \geq 2 \). From (21) and (22), we get
\[
1 = 5y - (x - ky)(5a + 3) = 5[y - (x - ky)a] - 3(x - ky), \quad (25)
\]
\[
1 = (x - ky)(5a + 3) - 5y = 3(x - ky) - 5[y - (x - ky)a]. \quad (26)
\]
The minimum solution is obtained from (27) as follows:
\[ y - (x - ky)a = 1, \quad x - ky = 2 \implies y = 2a + 1 \quad \text{and} \quad x = k(2a + 1) + 2. \]
Hence, in this case, the minimum \( m \) in (12) is \( m = qy = q(2a + 1) \).

Case 4: When \( p \) is of the form \( p = 5a + 4 \) for some integer \( a \geq 2 \).
In this case, 5 divides \( p + 1 \). Then, the minimum solution is obtained from (21), which is
\[ y = \frac{p+1}{5} \quad \text{(and} \quad x - ky = 1). \]
Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{2(p+1)}{5} - 1 \).

**Corollary 2.10.** Let \( p \) and \( q \) be two primes with \( q > p \geq 11 \). Let \( q = (k+1)p - 5 \) for some integer \( k \geq 1 \). Then,
\[
Z(pq) = \begin{cases} 
q(p-1), & \text{if } 5|(p-1); \\
q(2a+1), & \text{if } p = 5a + 2; \\
q(2a+1) - 1, & \text{if } p = 5a + 3; \\
\frac{q(p+1)}{5}, & \text{if } 5|(p+1).
\end{cases}
\]

**Proof.** From (2.1) and (2.2) with \( \ell = p - 5 \), we have respectively
\[
[(k+1)y - x]p - 5y = 1, \quad (27)
\]
\[
5y - [(k+1)y - x]p = 1. \quad (28)
\]
As in the proof of Corollary 2.9, we consider the following four possibilities:

Case 1: When \( p \) is of the form \( p = 5a + 1 \) for some integer \( a \geq 2 \).
In this case, 5 divides \( p - 1 \). Then, the minimum solution is obtained from (27), which is
\[ y = \frac{p-1}{5} \quad \text{(and} \quad x = (k+1)y - 1). \]
Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{2(p-1)}{5} - 1 \).

Case 2: When \( p \) is of the form \( p = 5a + 2 \) for some integer \( a \geq 2 \).
In this case, from (27) and (28), we get respectively
\[
1 = [(k+1)y - x](5a + 2) - 5y = 2[(k+1)y - x] - 5[y - a(k + 1)y - x], \quad (29)
\]
\[
1 = 5y - [(k+1)y - x](5a + 2) = 5[y - a(k + 1)y - x] - 2[(k+1)y - x]. \quad (30)
\]
Clearly, the minimum solution is obtained from (30), which is
\[ y - a(k + 1)y - x = 1, (k + 1)y - x = 2 \implies y = 2a + 1 \quad \text{and} \quad x = (k + 1)(2a + 1) - 2. \]
Hence, in this case, the minimum \( m \) in (12) is \( m = qy = q(2a + 1) \).

Case 3: When \( p \) is of the form \( p = 5a + 3 \) for some integer \( a \geq 2 \).
In this case, from (27) and (28), we get respectively
\[ 1 = [(k + 1)y - x](5a + 3) - 5y = 3[(k + 1)y - x] - 5[y - a(k + 1)y - x], \quad (31) \]
\[ 1 = 5y - [(k + 1)y - x](5a + 3) = 5y - a(k + 1)y - x - 3[(k + 1)y - x]. \quad (32) \]
The minimum solution is obtained from (31) as follows:
\[ y - a(k + 1)y - x = 1, \quad (33) \]
\[ (x - ky)p - 6y = 1. \quad (34) \]

Now, for any prime \( p \geq 13 \), exactly one of the following two cases can occur: Either \( p - 1 \) is divisible by 6, or \( p + 1 \) is divisible by 6. We thus consider the two possibilities separately below:

**Case 1:** When 6 divides \( p - 1 \).
In this case, the minimum solution, obtained from (34), is \( y = \frac{p-1}{6} \) (and \( x = ky + 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy = \frac{q(p-1)}{6} \).

**Case 2:** When 6 divides \( p + 1 \).
In this case, (33) gives the minimum solution, which is \( y = \frac{p+1}{6} \) (and \( x = ky+1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{q(p+1)}{6} - 1 \).

**Corollary 2.11.** Let \( p \) and \( q \) be two primes with \( q > p \geq 13 \). Let \( q = kp + 6 \) for some integer \( k \geq 1 \). Then,
\[
Z(pq) = \begin{cases} 
\frac{q(p-1)}{6}, & \text{if } 6|(p-1); \\
\frac{q(p+1)}{6} - 1, & \text{if } 6|(p+1). 
\end{cases}
\]

**Proof.** From (2.1) and (2.2) with \( \ell = 6 \), we have respectively
\[ 6y - (x - ky)p = 1, \quad (35) \]
\[ (x - ky)p - 6y = 1. \quad (36) \]

Now, for any prime \( p \geq 13 \), exactly one of the following two cases can occur: Either \( p - 1 \) is divisible by 6, or \( p + 1 \) is divisible by 6. We thus consider the two possibilities separately below:

**Case 1:** When 6 divides \( p - 1 \).
In this case, the minimum solution, obtained from (34), is \( y = \frac{p-1}{6} \) (and \( x = ky + 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy = \frac{q(p-1)}{6} \).

**Case 2:** When 6 divides \( p + 1 \).
In this case, (33) gives the minimum solution, which is \( y = \frac{p+1}{6} \) (and \( x = ky+1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{q(p+1)}{6} - 1 \).

**Corollary 2.12.** Let \( p \) and \( q \) be two primes with \( q > p \geq 13 \). Let \( q = (k+1)p - 6 \) for some integer \( k \geq 1 \). Then,
\[
Z(pq) = \begin{cases} 
\frac{q(p+1)}{6}, & \text{if } 6|(p+1); \\
\frac{q(p-1)}{6} - 1, & \text{if } 6|(p-1). 
\end{cases}
\]

**Proof.** From (2.1) and (2.2) with \( \ell = p - 6 \), we have respectively
\[ [(k + 1)y - x](p - 6y) = 1, \quad (35) \]
\[ 6y - [(k + 1)y - x]p = 1. \quad (36) \]

We now consider the following two cases which are the only possibilities (as noted in the proof of Corollary 2.11):

**Case 1:** When 6 divides \( p + 1 \).
In this case, the minimum solution, obtained from (36), is $y = \frac{x + 1}{6}$ (and $x = (k + 1)y - 1$). Therefore, the minimum $m$ in (12) is $m = qy = \frac{2(p - 1)}{6}$.

Case 2: When 6 divides $p - 1$.

Here, the minimum solution is obtained from (35), which is $y = \frac{p - 1}{6}$ (and $x = (k + 1)y - 1$). Therefore, the minimum $m$ in (12) is $m = qy = \frac{2(p - 1)}{6} - 1$.

**Corollary 2.13.** Let $p$ and $q$ be two primes with $q > p \geq 13$. Let $q = kp + 7$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{2(p - 1)}{7}, & \text{if } 7(p - 1); \\ q(3a + 1) - 1, & \text{if } p = 7a + 2; \\ q(2a + 1) - 1, & \text{if } p = 7a + 3; \\ q(2a + 1), & \text{if } p = 7a + 4; \\ q(3a + 2), & \text{if } p = 7a + 5; \\ \frac{q(p + 1)}{7} - 1, & \text{if } 7(p + 1). \end{cases}$$

**Proof.** From (2.1) and (2.2) with $\ell = 7$, we have respectively

$$7y - (x - ky)p = 1, \quad (37)$$

$$(x - ky)p - 7y = 1. \quad (38)$$

Now, for any prime $p \geq 11$, exactly one of the following six cases occur:

Case 1: When $p$ is of the form $p = 7a + 1$ for some integer $a \geq 2$.

In this case, 7 divides $p - 1$. Then, the minimum solution is obtained from (38), which is $y = \frac{p - 1}{7}$ (and $x - ky = 1$).

Therefore, the minimum $m$ in (12) is $m = qy = \frac{2(p - 1)}{7}$.

Case 2: When $p$ is of the form $p = 7a + 2$ for some integer $a \geq 2$.

In this case, from (37) and (38), we get respectively

$$1 = 7y - (x - ky)(7a + 2) = 7[y - (x - ky)a] - 2(x - ky), \quad (39)$$

$$1 = (x - ky)(7a + 2) - 7y = 2(x - ky) - 7[y - (x - ky)a]. \quad (40)$$

Clearly, the minimum solution is obtained from (39), which is

$$y - (x - ky)a = 1, x - ky = 3 \implies y = 3a + 1 \text{ (and } x = k(3a + 1) + 3).$$

Hence, in this case, the minimum $m$ in (12) is $m = qy - 1 = q(3a + 1) - 1$.

Case 3: When $p$ is of the form $p = 7a + 3$ for some integer $a \geq 2$. Here, from (37) and (38),

$$1 = 7y - (x - ky)(7a + 3) = 7[y - (x - ky)a] - 3(x - ky), \quad (41)$$

$$1 = (x - ky)(7a + 3) - 7y = 3(x - ky) - 7[y - (x - ky)a]. \quad (42)$$

The minimum solution is obtained from (41) as follows:

$$y - (x - ky)a = 1, x - ky = 2 \implies y = 2a + 1 \text{ (and } x = k(2a + 1) + 2).$$

Hence, in this case, the minimum $m$ in (12) is $m = qy - 1 = q(2a + 1) - 1$. 

```
Case 4 : When \( p \) is of the form \( p = 7a + 4 \) for some integer \( a \geq 2 \). In this case, from (37) and (38), we get respectively

\[
1 = 7y - (x - ky)(7a + 4) = 7[y - (x - ky)a] - 4(x - ky),
\]

\[
1 = (x - ky)(7a + 4) - 7y = 4(x - ky) - 7[y - (x - ky)a].
\]

Clearly, the minimum solution is obtained from (44), which is

\[
y - (x - ky)a = 1, x - ky = 2 \implies y = 2a + 1 (\text{and} \ x = k(2a + 1) + 2).
\]

Hence, in this case, the minimum \( m \) in (12) is \( m = qy - 1 = q(2a + 1) + 1 \).

Case 5 : When \( p \) is of the form \( p = 7a + 5 \) for some integer \( a \geq 2 \). From (37) and (38), we have

\[
1 = 7y - (x - ky)(7a + 5) = 7[y - (x - ky)a] - 5(x - ky),
\]

\[
1 = (x - ky)(7a + 5) - 7y = 5(x - ky) - 7[y - (x - ky)a].
\]

The minimum solution is obtained from (46), which is

\[
y - (x - ky)a = 2, x - ky = 3 \implies y = 3a + 2 (\text{and} \ x = k(3a + 2) + 3).
\]

Hence, in this case, the minimum \( m \) in (12) is \( m = qy - 1 = q(3a + 2) \).

Case 6 : When \( p \) is of the form \( p = 7a + 6 \) for some integer \( a \geq 2 \). In this case, 7 divides \( p + 1 \). Then, the minimum solution is obtained from (37), which is \( y = \frac{p+1}{7} \) (and \( x - ky = 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{q(p+1)}{7} - 1 \).

**Corollary 2.14.** Let \( p \) and \( q \) be two primes with \( q > p \geq 13 \). Let \( q = (k+1)p - 7 \) for some integer \( k \geq 1 \).

Then,

\[
Z(pq) = \begin{cases} 
\frac{q(p-1)}{7}, & \text{if} \ 7|(p-1); \\
q(3a+1), & \text{if} \ p = 7a + 2; \\
q(2a+1), & \text{if} \ p = 7a + 3; \\
q(2a+1) - 1, & \text{if} \ p = 7a + 4; \\
q(3a+2) - 1, & \text{if} \ p = 7a + 5; \\
\frac{q(p+1)}{7}, & \text{if} \ 7|(p+1).
\end{cases}
\]

**Proof.** From (2.1) and (2.2) with \( \ell = 7 \), we have respectively

\[
[(k + 1)y - x]p - 7y = 1, \quad (47)
\]

\[
7y - [(k + 1)y - x]p = 1. \quad (48)
\]

We now consider the following six possibilities:

Case 1 : When \( p \) is of the form \( p = 7a + 1 \) for some integer \( a \geq 2 \). In this case, 7 divides \( p - 1 \). Then, the minimum solution is obtained from (47), which is \( y = \frac{p-1}{7} \) (and \( x = (k+1)y - 1 \)). Therefore, the minimum \( m \) in (12) is \( m = qy - 1 = \frac{q(p-1)}{7} - 1 \).

Case 2 : When \( p \) is of the form \( p = 7a + 2 \) for some integer \( a \geq 2 \). In this case, from (47) and (48), we get respectively

\[
1 = [(k + 1)y - x](7a + 2) - 7y = 2[(k + 1)y - x] - 7[y - a((k + 1)y - x)], \quad (49)
\]
Clearly, the minimum solution is obtained from (50), which is

$$y - a((k + 1)y - x) = 1, (k + 1)y - x = 3 \implies y = 3a + 1 \text{ (and } x = (k + 1)(3a + 1) - 3).$$

Hence, in this case, the minimum \( m \) in (12) is \( m = qy = q(3a + 1) \).

Case 3: When \( p \) is of the form \( p = 7a + 3 \) for some integer \( a \geq 2 \).

Here, from (47) and (48),

$$1 = [(k + 1)y - x](7a + 3) - 7y = 3[(k + 1)y - x] - 7[y - a((k + 1)y - x)],$$

(51)

$$1 = 7y - [(k + 1)y - x](7a + 3) = 7[y - a((k + 1)y - x)] - 3[(k + 1)y - x].$$

(52)

Then, (52) gives the minimum solution, which is:

$$y - a((k + 1)y - x) = 1, (k + 1)y - x = 2 \implies y = 2a + 1 \text{ (and } x = (k + 1)(2a + 1) - 2).$$

Hence, in this case, the minimum \( m \) in (12) is \( m = qy = q(2a + 1) \).

Case 4: When \( p \) is of the form \( p = 7a + 4 \) for some integer \( a \geq 2 \).

Here, from (47) and (48),

$$1 = [(k + 1)y - x](7a + 4) - 7y = 4[(k + 1)y - x] - 7[y - a((k + 1)y - x)],$$

(53)

$$1 = 7y - [(k + 1)y - x](7a + 4) = 7[y - a((k + 1)y - x)] - 4[(k + 1)y - x].$$

(54)

Clearly, the minimum solution is obtained from (53) as follows:

$$y - a((k + 1)y - x) = 1, (k + 1)y - x = 2 \implies y = 2a + 1 \text{ (and } x = (k + 1)(2a + 1) - 2).$$

Hence, in this case, the minimum \( m \) in (12) is \( m = qy - 1 = q(2a + 1) - 1 \).

Case 5: When \( p \) is of the form \( p = 7a + 5 \) for some integer \( a \geq 2 \).

In this case, from (47) and (48), we get respectively

$$1 = [(k + 1)y - x](7a + 5) - 7y = 5[(k + 1)y - x] - 7[y - a((k + 1)y - x)],$$

(55)

$$1 = 7y - [(k + 1)y - x](7a + 5) = 7[y - a((k + 1)y - x)] - 5[(k + 1)y - x].$$

(56)

Then, (55) gives the following minimum solution:

$$y - a((k + 1)y - x) = 2, (k + 1)y - x = 3 \implies y = 3a + 2 \text{ (and } x = (k + 1)(3a + 2) - 3).$$

Hence, in this case, the minimum \( m \) in (12) is \( m = qy - 1 = q(3a + 2) - 1 \).

Case 6: When \( p \) is of the form \( p = 7a + 6 \) for some integer \( a \geq 2 \). In this case, \( 7 \) divides \( p + 1 \).

Then, the minimum solution is obtained from (48), which is \( y = \frac{p + 1}{7} \) (and \( x = (k + 1)y - 1 \)).

Therefore, the minimum \( m \) in (12) is \( m = qy = \frac{q(p + 1)}{7} \).

**Corollary 2.15.** Let \( p \) and \( q \) be two primes with \( q > p \geq 13 \). Let \( q = kp + 8 \) for some integer \( k \geq 1 \).

Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{8}, & \text{if } 8|(p-1); \\ q(3a + 1), & \text{if } p = 8a + 3; \\ q(3a + 2) - 1, & \text{if } p = 8a + 5; \\ \frac{q(p+1)}{8} - 1, & \text{if } 8|(p+1). \end{cases}$$
Proof. From (2.1) and (2.2) with $\ell = 8$, we have respectively

$$8y - (x - ky)p = 1, \quad (57)$$

$$x - ky)p - 8y = 1. \quad (58)$$

Now, for any prime $p \geq 13$, exactly one of the following four cases occur:

Case 1: When $p$ is of the form $p = 8a + 1$ for some integer $a \geq 2$.

In this case, $8$ divides $p - 1$. Then, the minimum solution is obtained from (58), which is $y = \frac{p-1}{8}$ (and $x - ky = 1$).

Therefore, the minimum $m$ in (12) is $m = qy = \frac{2(p-1)}{8}$.

Case 2: When $p$ is of the form $p = 8a + 3$ for some integer $a \geq 2$.

In this case, from (57) and (58), we get respectively

$$1 = 8y - (x - ky)(8a + 3) = 8[y - (x - ky)a] - 3(x - ky), \quad (59)$$

$$1 = (x - ky)(8a + 3) - 8y = 3(x - ky) - 8[y - (x - ky)a]. \quad (60)$$

Clearly, the minimum solution is obtained from (60), which is

$y - (x - ky)a = 1, x - ky = 3 \implies y = 3a + 1$ (and $x = k(3a + 1) + 3$).

Hence, in this case, the minimum $m$ in (12) is $m = qy = q(3a + 1)$.

Case 3: When $p$ is of the form $p = 8a + 5$ for some integer $a \geq 2$.

From (57) and (58), we get

$$1 = 8y - (x - ky)(8a + 5) = 8[y - (x - ky)a] - 5(x - ky), \quad (61)$$

$$1 = (x - ky)(8a + 5) - 8y = 5(x - ky) - 8[y - (x - ky)a]. \quad (62)$$

The minimum solution is obtained from (61) as follows:

$y - (x - ky)a = 2, x - ky = 3 \implies y = 3a + 2$ (and $x = k(3a + 2) + 3$).

Hence, in this case, the minimum $m$ in (12) is $m = qy - 1 = q(3a + 2) - 1$.

Case 4: When $p$ is of the form $p = 8a + 7$ for some integer $a \geq 2$.

In this case, $8$ divides $p - 1$. Then, the minimum solution is obtained from (57), which is $y = \frac{p+1}{8}$ (and $x - ky = 1$). Therefore, the minimum $m$ in (12) is $m = qy - 1 = \frac{q(p+1)}{8} - 1$.

Corollary 2.16. Let $p$ and $q$ be two primes with $q > p \geq 13$. Let $q = (k + 1)p - 8$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} 
\frac{q(p-1)}{8}, & \text{if } 8|p-1; \\
q(3a + 1) - 1, & \text{if } p = 8a + 3; \\
q(3a + 2), & \text{if } p = 8a + 5; \\
\frac{q(p+1)}{8}, & \text{if } 8|(p+1). 
\end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = p - 8$, we have respectively

$$[(k + 1)y - x]p - 8y = 1, \quad (63)$$

$$8y - [(k + 1)y - x]p = 1. \quad (64)$$
We now consider the following two cases that may arise:

Case 1 : When \( p \) is of the form \( p = 8a + 1 \) for some integer \( a \geq 2 \).

In this case, \( 8 \) divides \( p - 1 \). Then, the minimum solution is obtained from (63), which is

\[ y = \frac{p - 1}{8} \quad \text{(and } x = (k + 1)y - 1 \text{). Therefore, the minimum } m \text{ in (12) is } m = qy - 1 = \frac{q(p - 1)}{8} - 1. \]

Case 2 : When \( p \) is of the form \( p = 8a + 3 \) for some integer \( a \geq 2 \).

In this case, from (63) and (64), we get respectively

\[ 1 = [(k + 1)y - x] + 8 = 2[(k + 1)y - x] - 8[y - a{(k + 1)y - x}], \quad (65) \]

\[ 1 = 8y - [(k + 1)y - x](8a + 3) = 8[y - a{(k + 1)y - x}] - 3[(k + 1)y - x]. \quad (66) \]

Clearly, the minimum solution is obtained from (65), which is

\[ y - a{(k + 1)y - x} = 1, (k + 1)y - x = 3 \implies y = 3a + 1 \text{ and } x = (k + 1)(3a + 1) - 3. \]

Hence, in this case, the minimum \( m \) in (12) is \( m = qy = q(3a + 1) - 1 \).

Case 3 : When \( p \) is of the form \( p = 8a + 5 \) for some integer \( a \geq 1 \).

In this case, from (63) and (64), we get respectively

\[ 1 = [(k + 1)y - x](8a + 5) - 8 = 5[(k + 1)y - x] - 8[y - a{(k + 1)y - x}], \quad (67) \]

\[ 1 = 8y - [(k + 1)y - x](8a + 5) = 8[y - a{(k + 1)y - x}] - 5[(k + 1)y - x]. \quad (68) \]

The minimum solution is obtained from (68) as follows:

\[ y - a{(k + 1)y - x} = 2, (k + 1)y - x = 3 \implies y = 3a + 2 \text{ and } x = (k + 1)(3a + 2) - 3. \]

Hence, in this case, the minimum \( m \) in (12) is \( m = qy = q(3a + 2) \).

Case 4 : When \( p \) is of the form \( p = 8a + 7 \) for some integer \( a \geq 2 \).

In this case, \( 8 \) divides \( p + 1 \). Then, the minimum solution is obtained from (64), which is

\[ y = \frac{p + 1}{8} \quad \text{(and } x = (k + 1)y - 1 \text{). Therefore, the minimum } m \text{ in (12) is } m = qy = \frac{q(p + 1)}{8}. \]

We now consider the case when \( n \) is a composite number. Let

\[ Z(n) = m_0 \text{ for some integer } m_0 \geq 1 \text{. Then, } n \text{ divides } \frac{m_0(m_0 + 1)}{2}. \]

We now consider the following two cases that may arise :

Case 1 : \( m_0 \) is even (so that \( m_0 + 1 \) is odd).

(1) Let \( n \) be even. In this case, \( n \) does not divide \( \frac{m_0}{2} \), for otherwise,

\[ \frac{n|m_0}{2} \implies \frac{n|m_0(m_0 + 1)}{2} \implies Z(n) \leq (m_0 - 1). \]

(2) Let \( n \) be odd. In such a case, \( n \) does not divide \( m_0 \).

Case 2 : \( m_0 \) is odd (so that \( m_0 + 1 \) is even).

(1) Let \( n \) be even. Then, \( n \) does not divide \( m_0 \).

(2) Let \( n \) be odd. Here, \( n \) does not divide \( m_0 \), for

\[ \frac{n|m_0}{2} \implies \frac{n|m_0(m_0 - 1)}{2} \implies Z(n) \leq (m_0 - 1). \]
Thus, if \( n \) is a composite number, \( n \) does not divide \( m_0 \).

Now let
\[
n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i} (p_{i+1} \cdots p_s)^{\alpha_s}
\]
be the representation of \( n \) in terms of its distinct prime factors \( p_1, p_2, \ldots, p_i, p_{i+1}, \ldots, p_s \), not necessarily ordered. Then, one of \( m_0 \) and \( m_0 + 1 \) is of the form
\[
2^\beta p_1^{\beta_1} p_2^{\beta_2} \cdots p_i^{\beta_i} q_{i+1} \cdots q_s^{\beta_s}
\]
for some \( 1 \leq i < s \); \( \beta_j \geq \alpha_j \) for \( 1 \leq j < i \), and the other one is of the form
\[
p_{i+1}^{\gamma_{i+1}} \cdots p_s^{\gamma_s} r_{s+1}^{\gamma_{s+1}} \cdots r_u^{\gamma_u} \geq \alpha_j
\]
for \( i + 1 \leq j < s \); where \( q_{i+1}, \ldots, q_s \) and \( r_{s+1}, \ldots, r_u \) are all distinct primes, not necessarily ordered.

§3. Some Observations

Some observations about the Pseudo-Smarandache Function are given below:

Remark 3.1. Kashihara raised the following questions (see Problem 7 in [1]):
(1) Is there any integer \( n \) such that \( Z(n) > Z(n + 1) > Z(n + 2) > Z(n + 3) \)?
(2) Is there any integer \( n \) such that \( Z(n) < Z(n + 1) < Z(n + 2) < Z(n + 3) \)?

The following examples answer the questions in the affirmative:

(1) \( Z(256) = 511 > 256 = Z(257) > Z(258) = 128 > 111 = Z(259) > Z(260) = 39 \).

(2) \( Z(159) = 53 < 64 = Z(160) < Z(161) = 69 < 80 = Z(162) < Z(163) = 162 \).

These examples show that even five consecutive increasing or decreasing terms are available in the sequence \( \{Z(n)\} \).

Remark 3.2 Kashihara raises the following question (see Problem 5 in [1]): Given any integer \( m_0 \geq 1 \), how many \( n \) are there such that \( Z(n) = m_0 \)?

Given any integer \( m_0 \backslash 3 \), let
\[
Z^{-1}(m_0) = \{ n : n \in N, Z(n) = m_0 \},
\]
with
\[
Z^{-1}(1) = \{ 1 \}, Z^{-1}(2) = \{ 3 \}.
\]

Thus, for example, \( Z^{-1}(8) = \{ 8, 12, 18, 36 \} \).

By Lemma 2.1,
\[
n_{\text{max}} = \frac{m_0(m_0 + 1)}{2} \in Z^{-1}(m_0).
\]

This shows that the set \( Z^{-1}(m_0) \) is non-empty; moreover, \( n_{\text{max}} \) is the biggest element of \( Z^{-1}(m_0) \), so that \( Z^{-1}(m_0) \) is also bounded. Clearly, \( n \in Z^{-1}(m_0) \) only if \( n \) divides \( f(m_0) \equiv \)
$m_0(m_0 + 1)/2$. This is a necessary condition, but is not sufficient. For example, $4|36 \equiv f(8)$ but $4 \not\in Z^{-1}(8)$. The reason is that $Z(n)$ is not bijective. Let

$$Z^{-1} = \sum_{m=1}^{\infty} Z^{-1}(m)$$

Let $n \in Z^{-1}$. Then, there is one and only one $m_0$ such that $n \in Z^{-1}(m_0)$, that is, there is one and only one $m_0$ such that $Z(n) = m_0$.

However, we have the following result whose proof is almost trivial: $n \in Z^{-1}(m_0)$ if and only if the following two conditions are satisfied

1. $n$ divides $m_0(m_0 + 1)/2$,
2. $n$ does not divide $m(m + 1)/2$ for any $m$ with $3 \leq m \leq m_0 - 1$.

Since $4|28 \equiv f(7)$, it therefore follows that $4 \not\in Z^{-1}(8)$.

Given any integer $m_0 \geq 1$, let $C(m_0)$ be the number of integers $n$ such that $Z(n) = m_0$, that is, $C(m_0)$ denotes the number of elements of $Z^{-1}(m_0)$. Then,

$$1 \leq C(m_0) \leq d(m_0(m_0 + 1)/2) - 1 \text{ for } m_0 \geq 3; \quad C(1) = 1, \quad C(2) = 2,$$

where, for any integer $n$, $d(n)$ denotes the number of divisors of $n$ including 1 and $n$. Now, let $p \geq 3$ be a prime. Since, by Lemma 1.2, $Z(p) = p - 1$, we see that $p \in Z^{-1}(p - 1)$ for all $p \geq 3$.

Let $n \in Z^{-1}(p - 1)$. Then, $n$ divides $p(p - 1)/2$. This shows that $n$ must divide $p$, for otherwise

$$n|\frac{p - 1}{2} \Rightarrow n|\frac{(p - 1)(p - 2)}{2} \Rightarrow Z(n) \leq p - 2,$$

contradicting the assumption. Thus, any element of $Z^{-1}(p - 1)$ is a multiple of $p$. In particular, $p$ is the minimum element of $Z^{-1}(p - 1)$. Thus, if $p \geq 5$ is a prime, then $Z^{-1}(p - 1)$ contains at least two elements, namely, $p$ and $p(p - 1)/2$.

Next, let $p$ be a prime factor of $m_0(m_0 + 1)/2$. Since, by Lemma 1.2, $Z(p) = p - 1$, we see that $p \in Z^{-1}(m_0)$ if and only if $p - 1 \geq m_0$, that is, if and only if $p \geq m_0 + 1$.

**Remark 3.3.** Ibstedt[2] provides a table of values of $Z(n)$ for $1 \leq n \leq 1000$. A closer look at these values reveal some facts about the values of $Z(n)$. These observations are given in the conjectures below, followed by discussions in each case.

**Conjecture 1.** $Z(n) = 2n - 1$ if and only if $n = 2^k$ for some integer $k \geq 0$.

Let, for some integer $n \geq 1$,

$$Z(n) = m_0, \text{ where } m_0 = 2n - 1.$$

Note that the conjecture is true for $n = 1$ (with $k = 0$). Also, note that $n$ must be composite. Now, since $m_0 = 2n - 1$, and since $n|\frac{m_0(m_0 + 1)}{2}$, it follows that $n$ does not divide $m_0$, and $n|\frac{m_0 + 1}{2}$; moreover, by virtue of the definition of $Z(n)$, $n$ does not divide $m_0$, and $n|\frac{m + 1}{2}$ for all $1 \leq m \leq m_0 - 1$.

Let

$$Z(2n) = m_1.$$

We want to show that $m_1 = 2m_0 + 1$. Since $n|\frac{m_0 + 1}{2}$, it follows that $2n|\frac{2(m_0 + 1)}{2} = \frac{(2m_0 + 1)}{2}$; moreover, $2n$ does not divide

$$\frac{2(m + 1)}{2} = \frac{(2m + 1)}{2}.$$
for all $1 \leq m \leq m_0 - 1$.

Thus,
\[
m_1 = 2m_0 + 1 = 2(2n - 1) + 1 = 2^2n - 1.
\]

All these show that
\[
Z(n) = 2n - 1 \Rightarrow Z(2n) = 2^2n - 1.
\]

Continuing this argument, we see that
\[
Z(n) = 2n - 1 \Rightarrow Z(2^k n) = 2^{k+1}n - 1.
\]

Since $Z(1) = 1$, it then follows that $Z(2^k) = 2^{k+1} - 1$.

**Conjecture 2.** $Z(n) = n - 1$ if and only if $n = p^k$ for some prime $p \geq 3$ and integer $k \geq 1$.

Let, for some integer $n \geq 2$,
\[
Z(n) = m_0, \text{ where } m_0 = n - 1.
\]

Then, $2|m_0$ and $n|(m_0 + 1)$; moreover, $n$ does not divide $m + 1$ for any $1 \leq m \leq m_0 - 1$.

Let
\[
Z(n^2) = m_1.
\]

Since $n|(m_0 + 1)$, it follows that
\[
n^2|(m_0 + 1)^2 = (m_0^2 + 2m_0) + 1;
\]

moreover, $n^2$ does not divide $(m + 1)^2 = (m^2 + 2m) + 1$ for all $1 \leq m \leq m_0 - 1$.

Thus,
\[
m_1 = m_0^2 + 2m_0 = (n - 1)^2 + 2(n - 1) = n^2 - 1,
\]

so that (since $2|m_0 \Rightarrow 2|m_1$)
\[
Z(n) = n - 1 \Rightarrow Z(n^2) = n^2 - 1.
\]

Continuing this argument, we see that
\[
Z(n) = n - 1 \Rightarrow Z(n^{2k}) = n^{2k} - 1.
\]

Next, let
\[
Z(n^{2k+1}) = m_2 \text{ for some integer } k \geq 1.
\]

Since $n|(m_0 + 1)$, it follows that
\[
n^{2k+1}|(m_0 + 1)^{2k+1} = [(m_0 + 1)^{2k+1} - 1] + 1;
\]

moreover,
\[
n^{2k+1} \text{ does not divide }\]
\[
[(m + 1)^{2k+1} = [(m + 1)^{2k+1} - 1] + 1
\]

for all $1 \leq m \leq m_0 - 1$. Thus,
\[
m_2 = (m_0 + 1)^{2k+1} - 1 = n^{2k+1} - 1,
\]
so that (since $2 | m_0 \Rightarrow 2 | m_2$)
\[ Z(n) = n - 1 \Rightarrow Z(n^{2k+1}) = n^{2k+1} - 1. \]

All these show that
\[ Z(n) = n - 1 \Rightarrow Z(n^k) = n^k - 1. \]

Finally, since $Z(p) = p - 1$ for any prime $p \geq 3$, it follows that $Z(p^k) = p^k - 1$.

**Conjecture 3.** If $n$ is not of the form $2^k$ for some integer $k \geq 0$, then $Z(n) < n$. First note that, we can exclude the possibility that $Z(n) = n$, because
\[ n | n(n + 1) \Rightarrow n | n(n - 1) \Rightarrow Z(n) \leq n - 1. \]

So, let
\[ Z(n) = m_0 \text{ with } m_0 > n. \]

Note that, $n$ must be a composite number, not of the form $p^k$ ($p \geq 3$ is prime, $k \geq 0$). Let
\[ m_0 = an + b \text{ for some integers } a \geq 1, 1 \leq b \leq n. \]

Then,
\[ m_0(m_0 + 1) = (an + b)(an + b + 1) = n(a^2n + 2ab + a) + b(b + 1). \]

Therefore,
\[ n | m_0(m_0 + 1) \text{ if and only if } b + 1 = n. \]

But, by Conjecture 1, $b + 1 = n$ leads to the case when $n$ is of the form $2^k$.

**Remark 3.4.** Kashihara proposes (see Problem 4(a) in [1]) to find all the values of $n$ such that $Z(n) = Z(n + 1)$. In this connection, we make the following conjecture:

**Conjecture 4.** For any integer $n \geq 1$, $Z(n) \neq Z(n + 1)$. Let
\[ Z(n) = Z(n + 1) = m_0 \text{ for some } n \in N, m_0 \geq 1. \]

Then, neither $n$ nor $n + 1$ is a prime.

To prove this, let $n = p$, where $p$ is a prime. Then, by Lemma 1.2, $Z(n) = Z(p) = p - 1$.
\[ n + 1 = p + 1 \text{ does not divide } \frac{p(p - 1)}{2} \Rightarrow Z(n + 1) \neq p - 1 = Z(n). \]

Similarly, it can be shown that $n + 1$ is not a prime. Thus, both $n$ and $n + 1$ are composite numbers.

From (68), we see that both $n$ and $n + 1$ divide $m_0(m_0 + 1)/2$. Let
\[ \frac{m_0(m_0 + 1)}{2} = an \text{ for some integer } a \geq 1. \]

Since $n + 1$ divides $m_0(m_0 + 1)$ and since $n + 1$ does not divide $n$, it follows that $n + 1$ must divide $a$. So, let
\[ a = b(n + 1) \text{ for some integer } b \geq 1. \]
Then, 
\[
\frac{m_0(m_0 + 1)}{2} = ab(n + 1),
\]
which shows that 
\[
n(n + 1) \text{ must divide } \frac{m_0(m_0 + 1)}{2}.
\] (70)
From (69), we see that 
\[
Z(n(n + 1)) \leq m_0,
\]
which, together with Lemma 1.5 (that \(Z(n(n + 1)) \geq Z(n))\), gives
\[
Z(n(n + 1)) = m_0.
\] (71)
From (70), we see that 
\[
n(n + 1) \frac{m_0(m_0 + 1)}{2} \Rightarrow n(n + 1) \frac{m_0 + 1}{2} \Rightarrow Z\left(\frac{n(n + 1)}{2}\right) \leq m_0.
\]
Thus, by virtue of Lemma 2.1, 
\[
Z\left(\frac{n(n + 1)}{2}\right) = n \leq m_0 = Z(n).
\] It can easily be verified that neither \(n\) nor \(n + 1\) can be of the form \(2k\). Thus, if Conjecture 3 is true then Conjecture 4 is also true.

**Remark 3.5.** An integer \(n > 0\) is called \(f\)-perfect if
\[
n = \sum_{i=1}^{k} f(d_i),
\]
where \(d_1, d_2, \ldots, d_k\) are the proper divisors of \(n\), and \(f\) is an arithmetical function. In particular, \(n\) is Pseudo-Smarandache perfect if
\[
n = \sum_{i=1}^{k} Z(d_i).
\]
In [4], Ashbacher reports that the only Pseudo-Smarandache perfect numbers less than 1,000,000 are \(n = 4, 6, 471544\). However, since \(n = 471544\) is of the form \(n = 8p\) with \(p = 58943\), its only perfect divisors are 1, 2, 4, 8, \(p\), 2\(p\) and 4\(p\). Since \(8(p + 1) = 58944\), it follows from Lemma 1.2, Theorem 2.1 and Theorem 2.7 that
\[
Z(p) = p - 1, \quad Z(2p) = p, \quad Z(4p) = p,
\]
so that
\[
n = 471544 > \sum_{i=1}^{k} Z(d_i),
\]
so that \(n = 471544\) is not Pseudo-Smarandache perfect.
References


