The relationship between $S_p(n)$ and $S_p(kn)$

Weiyi Zhu

College of Mathematics, Physics and Information Engineer, Zhejiang Normal University  
Jinhua, Zhejiang, P.R.China

Abstract For any positive integer $n$, let $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by $p^n$, where $p$ be a prime. The main purpose of this paper is using the elementary methods to study the relationship between $S_p(n)$ and $S_p(kn)$, and give an interesting identity.

Keywords The primitive numbers of power $p$, properties, identity

§1. Introduction and Results

Let $p$ be a prime and $n$ be any positive integer. Then we define the primitive numbers of power $p$ ($p$ be a prime) $S_p(n)$ as the smallest positive integer such that $m!$ is divided by $p^n$. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = S_3(4) = 9$, $\cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Zhang Wenpeng and Liu Duansen [3] had studied the asymptotic properties of $S_p(n)$, and obtained an interesting asymptotic formula for it. That is, for any fixed prime $p$ and any positive integer $n$, they proved that

$$S_p(n) = (p - 1)n + O \left( \frac{p}{\ln p} \ln n \right).$$

Yi Yuan [4] had studied the asymptotic property of $S_p(n)$ in the form $\frac{1}{p} \sum_{n \leq x} |S_p(n + 1) - S_p(n)|$, and obtained the following result: for any real number $x \geq 2$, let $p$ be a prime and $n$ be any positive integer,

$$\frac{1}{p} \sum_{n \leq x} |S_p(n + 1) - S_p(n)| = x \left( 1 - \frac{1}{p} \right) + O \left( \frac{\ln x}{\ln p} \right).$$

Xu Zhefeng [5] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, for any prime $p$ and complex number $s$ with $\text{Re } s > 1$, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^n(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.
And, let $p$ be a fixed prime, then for any real number $x \geq 1$ he got
\[
\sum_{n=1}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1} \left( \ln x + \gamma + \frac{p \ln p}{p-1} \right) + O(x^{-\frac{1}{2} + \varepsilon}),
\]
where $\gamma$ is the Euler constant, $\varepsilon$ denotes any fixed positive number.

Chen Guohui [7] had studied the calculation problem of the special value of famous Smaran-
dache function $S(n) = \min\{m : n \in N, n|m!\}$. That is, let $p$ be a prime and $k$ an integer with
$1 \leq k < p$. Then for polynomial $f(x) = x^{n_k} + x^{n_{k-1}} + \cdots + x^{n_1}$ with $n_k > n_{k-1} > \cdots > n_1$, we have:
\[
S(p^{f(p)}) = (p - 1)f(p) + pf(1).
\]
And, let $p$ be a prime and $k$ an integer with $1 \leq k < p$, for any positive integer $n$, we have:
\[
S \left( p^{kn} \right) = k \left( \phi(p^n) + \frac{1}{k} \right) p,
\]
where $\phi(n)$ is the Euler function. All these two conclusions above also hold for primitive function
$S_p(n)$ of power $p$.

In this paper, we shall use the elementary methods to study the relationships between
$S_p(n)$ and $S_p(kn)$, and get some interesting identities. That is, we shall prove the following:

**Theorem.** Let $p$ be a prime. Then for any positive integers $n$ and $k$ with $1 \leq n \leq p$ and
$1 < k < p$, we have the identities:
\[
S_p(kn) = kS_p(n), \quad \text{if } 1 < kn < p;
\]
\[
S_p(kn) = kS_p(n) - p \left\lfloor \frac{kn}{p} \right\rfloor, \quad \text{if } p < kn < p^2,
\]
where $[x]$ denotes the integer part of $x$.

§2. Two simple Lemmas

To complete the proof of the theorem, we need two simple lemmas which stated as following:

**Lemma 1.** For any prime $p$ and any positive integer $2 \leq l \leq p - 1$, we have:
(1) $S_p(n) = np$, if $1 \leq n \leq p$;
(2) $S_p(n) = (n - l + 1)p$, if $(l - 1)p + l - 2 < n \leq lp + l - 1$.

**Proof.** First we prove the case (1) of Lemma 1. From the definition of $S_p(n) = \min\{m : p^m|m!\}$, we know that to prove the case (1) of Lemma 1, we only to prove that $p^n||(np)!$. That is,
$p^n|(np)!$ and $p^{n+1}||(np)!$. According to Theorem 1.7.2 of [6] we only to prove that
$\sum_{j=1}^{\infty} \left\lfloor \frac{np}{p^j} \right\rfloor = n$.

In fact, if $1 \leq n < p$, note that $\left\lfloor \frac{n}{p^i} \right\rfloor = 0, \; i = 1, 2, \cdots$, we have
\[
\sum_{j=1}^{\infty} \left\lfloor \frac{np}{p^j} \right\rfloor = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^{j-1}} \right\rfloor = n + \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots = n.
\]
This means $S_p(n) = np$. If $n = p$, then $\sum_{j=1}^{\infty} \left\lfloor \frac{np}{p^j} \right\rfloor = n + 1$, but $p^p \nmid (p^2 - 1)!$ and $p^p|p^2!$. This prove the case (1) of Lemma 1. Now we prove the case (2) of Lemma 1. Using the same method
of proving the case (1) of Lemma 1 we can deduce that if \((l-1)p + l - 2 < n \leq lp + l - 1\), then\[
\left[ \frac{n-l+1}{p} \right] = l - 1, \quad \left[ \frac{n-l+1}{p^i} \right] = 0, \quad i = 2, 3, \ldots .
\]

So we have\[
\sum_{j=1}^{\infty} \left( \frac{(n-l+1)p}{p^j} \right) = \sum_{j=1}^{\infty} \left[ \frac{n-l+1}{p^j-1} \right] = n - l + 1 + \left[ \frac{n-l+1}{p} \right] + \left[ \frac{n-l+1}{p^2} \right] + \cdots
\]

From Theorem 1.7.2 of reference [6] we know that if \((l-1)p + l - 2 < n \leq lp + l - 1\), then\(|p^n|((n-l+1)p)!\). That is, \(S_p(n) = (n-l+1)p\). This proves Lemma 1.

**Lemma 2.** For any prime \(p\), we have the identity \(S_p(n) = (n-p+1)p\), if \(p^2 - 2 < n \leq p^2\).

**Proof.** It is similar to Lemma 1, we only need to prove \(|p^n|((n-p+1)p)!\). Note that if \(p^2 - 2 < n \leq p^2\), then\[
\left[ \frac{n-p+1}{p} \right] = p - 1, \quad \left[ \frac{n-p+1}{p^i} \right] = 0, \quad i = 2, 3, \ldots .
\]

So we have\[
\sum_{j=1}^{\infty} \left( \frac{(n-p+1)p}{p^j} \right) = \sum_{j=1}^{\infty} \left[ \frac{n-p+1}{p^j-1} \right] = n - p + 1 + \left[ \frac{n-p+1}{p} \right] + \left[ \frac{n-p+1}{p^2} \right] + \cdots
\]

From Theorem 1.7.2 of [6] we know that if \(p^2 - 2 < n \leq p^2\), then\(|p^n|((n-p+1)p)!\). That is, \(S_p(n) = (n-p+1)p\). This completes the proof of Lemma 2.

§3. Proof of Theorem

In this section, we shall use above Lemmas to complete the proof of our theorem.

Since \(1 \leq n \leq p\) and \(1 < k < p\), therefore we deduce \(1 < kn < p^2\). We can divide \(1 < kn < p^2\) into three interval \(1 < kn < p, (m-1)p+m-2 < kn \leq mp+m-1 (m = 2, 3, \ldots, p-1)\) and \(p^2 - 2 < kn \leq p^2\). Here, we discuss above three interval of \(kn\) respectively:

i) If \(1 < kn < p\), from the case (1) of Lemma 1 we have\[
S_p(kn) = knp = kS_p(n).
\]

ii) If \((m-1)p+m-2 < kn \leq mp+m-1 (m = 2, 3, \ldots, p-1)\), then from the case (2) of Lemma 1 we have\[
S_p(kn) = (kn-m+1)p = knp - (m-1)p = kS_p(n) - (m-1)p.
\]

In fact, note that if \((m-1)p+m-2 < kn < mp+m-1 (m = 2, 3, \ldots, p-1)\), then\[
m-1 + \left[ \frac{m-2}{p} \right] < \left[ \frac{kn}{p} \right] < m + \left[ \frac{m-1}{p} \right].
\]

Hence, \(\left[ \frac{kn}{p} \right] = m-1\). If kn = mp + m - 1,
then \( \left\lfloor \frac{kn}{p} \right\rfloor = m \), but \( p^{mp+m-1} \mid ((mp + m - 1)p - 1)! \) and \( p^{mp+m-1} \mid (mp + m - 1)p)! \). So we immediately get

\[
S_p(kn) = kS_p(n) - p \left\lfloor \frac{kn}{p} \right\rfloor .
\]

iii) If \( p^2 - 2 < kn \leq p^2 \), from Lemma 2 we have

\[
S_p(kn) = (kn - p + 1)p = knp - (p - 1)p.
\]

Similarly, note that if \( p^2 - 2 < kn \leq p^2 \), then \( p - \left\lfloor \frac{2}{p} \right\rfloor < \left\lfloor \frac{kn}{p} \right\rfloor \leq p \). That is, \( \left\lfloor \frac{kn}{p} \right\rfloor = p - 1 \). So we may immediately get

\[
S_p(kn) = kS_p(n) - p \left\lfloor \frac{kn}{p} \right\rfloor .
\]

This completes the proof of our Theorem.

References