Nikodým-type theorems for lattice group-valued measures with respect to filter convergence

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Abstract

We present some convergence and boundedness theorem with respect to filter convergence for lattice group-valued measures, whose techniques are based on sliding hump arguments.

We give some new versions of Nikodým convergence, boundedness and Brooks-Jewett-type theorems with respect to filter convergence for lattice group-valued measures, defined on a σ-algebra of an abstract nonempty set, in which sliding hump-type techniques are used.

Let $R$ be a Dedekind complete $(\ell)$-group, $Q$ be a countable set and $\mathcal{F}$ be a filter of $Q$. A subset of $Q$ is $\mathcal{F}$-stationary iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of $Q$.

A filter $\mathcal{F}$ of $Q$ is said to be diagonal iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$. Given an infinite set $I \subset Q$, a blocking of $I$ is a countable partition $\{D_k : k \in \mathbb{N}\}$ of $I$ into nonempty finite subsets.

A filter $\mathcal{F}$ of $Q$ is said to be block-respecting iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of $I$ there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\sharp$ denotes the number of elements of the set into brackets.

If $I \in \mathcal{F}^*$, then the trace $\mathcal{F}(I)$ of $\mathcal{F}$ on $I$ is the family $\{A \cap I : A \in \mathcal{F}\}$. It is not difficult to see that $\mathcal{F}(I)$ is a filter of $I$.

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2010 A.M.S. Subject Classifications: Primary: 26E50, 28A12, 28A33, 28B10, 28B15, 40A35, 46G10, 54A20, 54A40.
Secondary: 06F15, 06F20, 06F30, 22A10, 28A05, 40G15, 46G12, 54H11, 54H12.
Key words and phrases: lattice group, (free) filter, (s)-bounded measure, σ-additive measure, diagonal filter, block-respecting filter, limit theorem, Nikodým boundedness theorem, Stone Isomorphism technique.
Theorem 0.2. Let \( \mathcal{F} \) be a block-respecting filter of \( \mathbb{N} \), then \( \mathcal{F}(I) \) is a block-respecting filter of \( I \) for every \( I \in \mathcal{F}^* \).

Let \( \mathcal{F} \) be a filter of \( \mathbb{N} \). A sequence \( (x_n)_n \) in \( \mathbb{R} \) \((\mathcal{D}\mathcal{F})\)-converges to \( x \in \mathbb{R} \) iff there is a \((\mathcal{D})\)-sequence \((a_{t,l})_{t,l} \) with the property that \( \left\{ n \in \mathbb{N}: |x_n - x| \leq \sum_{t=1}^{\infty} a_{t_\varphi(t)} \right\} \in \mathcal{F} \) for each \( \varphi \in \mathbb{N}^\mathbb{N} \).

Let \( \Xi \) be any arbitrary nonempty set. A family \( \left( \beta_{\xi,n} \right)_{\xi \in \Xi, n \in \mathbb{N}} \) is said to be \((\mathcal{RD}\mathcal{F})\)-convergent to a family \( \left( \beta_\xi \right)_{\xi \in \Xi} \) with respect to \( \xi \in \Xi \) iff there is a regulator \((a_{t,l})_{t,l} \) such that for each \( \varphi \in \mathbb{N}^\mathbb{N} \) and \( \xi \in \Xi \) we get

\[
\left\{ n \in \mathbb{N} : |\beta_{\xi,n} - \beta_\xi| \leq \sum_{t=1}^{\infty} a_{t_\varphi(t)} \right\} \in \mathcal{F}.
\]

Given \( a < b \in \mathbb{R} \), set \( [a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \). For \( A, B \subset \mathbb{R} \), \( n \in \mathbb{N} \), put \( A + B = \{ a + b : a \in A, b \in B \} \), \( nA = \{ a + \ldots + a \} \) \((n \text{ times}) \). Let \( U_n = [-u_n, u_n] \), \( n \in \mathbb{N} \), be such that \( 0 < u_n \leq u_{n+1} \) for every \( n \in \mathbb{N} \). A set \( \{ x_n : n \in \mathbb{N} \} \subset \mathbb{R} \) is said to be \((\mathcal{PR})\)-\( \mathcal{F} \)-bounded by \((U_n)_n \), if \( \{ n \in \mathbb{N} : x_n \in U_n \} \in \mathcal{F} \), and \((\mathcal{PR})\)-eventually bounded by \((U_n)_n \) iff it is \((\mathcal{PR})\)\( \mathcal{F}_{\text{cofin}} \)-bounded by \((U_n)_n \).

We now give the main results.

**Theorem 0.1.** Let \( \mathbb{R} \) be a Dedekind complete \((\ell)\)-group, \( \mathcal{F} \) be a block-respecting filter of \( \mathbb{N} \), \( m_n : \Sigma \to \mathbb{R} \), \( n \in \mathbb{N} \), be a sequence of equibounded \( \sigma \)-additive measures, \( (C_k)_k \) be a disjoint sequence in \( \Sigma \), with

(i) \( (\mathcal{D}) \lim n m_n(C_k) = 0 \) for any \( k \in \mathbb{N} \), and

(ii) \( (\mathcal{RD}\mathcal{F}) \lim m_n(\bigcup_{k \in P} C_k) = 0 \) with respect to \( P \in \mathcal{P}(\mathbb{N}) \).

Then,

\[
(\mathcal{DF}) \lim n m_n(C_{k_n}) = 0; \tag{1}
\]

\( \gamma \gamma \) if \( \mathcal{F} \) is also diagonal and \( \mathbb{R} \) is super Dedekind complete and weakly \( \sigma \)-distributive, then the only condition \( (ii) \) is sufficient to get \( (1) \).

**Theorem 0.2.** Let \( \mathbb{R} \) be a Dedekind complete \((\ell)\)-group, \( (C_k)_k \) be as in Theorem 0.1, \( \mathcal{F} \) be a block-respecting filter of \( \mathbb{N} \), \( m_n : \Sigma \to \mathbb{R} \), \( n \in \mathbb{N} \), be an equibounded sequence of finitely additive measures, and assume that

(i) \( (\mathcal{D}) \lim n m_n(C_k) = 0 \) for any \( k \in \mathbb{N} \);

(ii) \( (\mathcal{RD}\mathcal{F}) \lim \sum_{k \in P} m_n(C_k) = 0 \) with respect to \( P \in \mathcal{P}(\mathbb{N}) \).

Then for every strictly increasing sequence \( (k_n)_n \) in \( \mathbb{N} \) we get

\[
(\mathcal{DF}) \lim n m_n(C_{k_n}) = 0. \tag{2}
\]
If $F$ is also diagonal and $R$ is super Dedekind complete and weakly $\sigma$-distributive, then the only condition (ii) is enough to get (2).

**Theorem 0.3.** Let $R$ be any Dedekind complete $(\ell)$-group, $u \in R$, $u > 0$, $U = [-u, u]$, $F$ be a block-respecting filter of $\mathbb{N}$, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of finitely additive measures, and assume that

0.3.1) for every disjoint sequence $(C_n)_n$ in $\Sigma$ and $j \in \mathbb{N}$ there is $Q_j \subset \mathbb{N}$ with $\sum_{n \in Q} m_j(C_n) \in U$ for each $Q \subset Q_j$.

Let $(C_n)_n$ be a disjoint sequence in $\Sigma$ and $(w_n)_n$ be an increasing sequence of positive elements of $R$. For each $n \in \mathbb{N}$, set $W_n := [-w_n, w_n]$ and $V_n := nW_n + U$. Moreover suppose that:

(i) the set $\{m_n(C_p) : n \in \mathbb{N}\}$ is $(PR)$-eventually bounded by $(W_n)_n$ for each $p \in \mathbb{N}$;
(ii) the set $\left\{\sum_{p \in P} m_j(C_p) : n \in \mathbb{N}\right\}$ is $(PR)$-$F$-bounded by $(W_n)_n$ for each $P \in \mathcal{P}(\mathbb{N})$.

Then we get:

(j) for every strictly increasing sequence $(l_n)_n$ in $\mathbb{N}$, the set $D := \{m_n(C_{l_n}) : n \in \mathbb{N}\}$ is $(PR)$-$F$-bounded by $(V_n)_n$;

(jj) if $F$ is also diagonal, then the only condition (ii) is enough in order that $D$ is $(PR)$-$F$-bounded by $(V_n)_n$. 
