The Arm Theory

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Abstract
We introduce the generalized Taylor formula which gives the development of function on an arbitrary chosen basis, which is called the Arm formula, and we give severals example of functions on non usual basis. Next, we generalize the Arm formula for functions which are not defined on zero-started basis, which is the shifted Arm formula.

Keywords : Taylor developpment, analysis

Introduction

The idea behind this article came to me when I was reading a mathematical article that deals with the Fourier theory. The Fourier series was constructed on the decomposition on the complex exponential basis with the inner product (or scalar product) :

$$< f, e^{ikx} > \ = \ \frac{1}{2\pi} \int_0^{2\pi} f(x)\ e^{-ikx}\ \mathrm{d}x \tag{0.1}$$

Next, I remembered the Taylor formula constructed on the decomposition on polynomial basis $\mathbb{C}[x] = \{1, x, x^2, ...\}$ with the scalar product

$$< f, x^k > \ = \ \lim_{x \to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k}\ f(x) \tag{0.2}$$

The decomposition of the geometric serie is well known with this formula (0.2), it is :

$$\frac{1}{1-x} \ = \ \sum_{k=0}^{\infty} \ x^k \tag{0.3}$$

or :

$$1 \ = \ (1-x)\ \ (1 + x + x^2 + x^3 + ...) \tag{0.4}$$

The number 1 in (0.4) is obtained in finding the number which, multiplied by the **<u>first</u>** term of $(1-x)$ (i.e. 1), will be one and the next terms of the series are obtained in deleting the previous term multiplyed with the second term of $(1-x)$.

Then I remarked that if you do the same thing with the **<u>second</u>** term instead of the first one, i.e. obtain 1 in finding the number which, multiplied by the second term of $(1-x)$ (i.e. x), will be one and the next terms of the series are obtained in deleting the previous term multiplyed with the first term of $(1-x)$. So with this method, we obtain :

$$1 \ = \ (1-x)\ \ (-x^{-1} - x^{-2} - x^{-3} - ...) \tag{0.5}$$

So you obtain an other decomposition of the geometric serie is

$$\frac{1}{1-x} \ = \ -\sum_{k=1}^{\infty} \ x^{-k} \tag{0.6}$$

This equation (0.6) is an other decomposition of the geometric serie on the inverse basis.

In that case, I searched the operator which go from the upper basis element x^{-k-1} to the lower basis element $(k+1)x^{-k}$. After a lot of difficulties, I found that this operator was

$$-x^2 \ \ \frac{\partial}{\partial x} \tag{0.7}$$

Again, after a lot of difficulties, I found in the same way the operator for the logarithm basis :

$$x\frac{\partial}{\partial x} \tag{0.8}$$

Therefore I generalized the operator which go from the upper basis element $u^{k+1}(x)$ to the lower basis element $(k+1)u^k(x)$:

$$\frac{\partial x}{\partial u}\ \frac{\partial}{\partial x} \tag{0.9}$$

So the scalar product generalized to each basis and at each point is given by

$$< f, \left(u(z) - z_0\right)^k >= \frac{1}{k!}\ \left[\lim_{z\to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^k f(z)\ \ \right] \tag{0.10}$$

In all this paper, the function $f(z)$ has a developpment on $\mathbb{C}[u(z) - z_0]$ only if

$$f(z) \in \mathcal{C}^{\infty}_{u(z)}(\mathcal{I}) \Leftrightarrow \left\{\forall z_0 \in \mathcal{I},\quad \forall k \in \mathbb{N},\ \lim_{z\to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^k f(z) < \infty\right\} \tag{0.11}$$

In all this paper, it is understood that every functions on natural basis ($u(x) = x$) do not converge when x tend to infinite. In the same way, the functions developped on the inverse basis ($u(x) = x^{-1}$) do not converge when x tend to 0.

In the first section of this article, we give and show the Arm formula which is the generalization of the Taylor formula for each basis $(u(z) - z_0)$. Here, we used the complex variable z in order to permit the decomposition on complex basis, for example (2.31). We also give this formula at each point z_0 such that $\exists z \ \mid\ u(z) = z_0$ in order to permit the developpment on basis which which doesn't reach 0, for example (2.37).

In the second section, we gives a lot of Arm decomposition (which is the Arm formula with a defined basis) on usual functions : powered functions, trigonometric function, hyperbolic function.

In the third section, we give the Arm formula for functions defined on positive and negative powers of the basis i.e. $\exists m \in \mathbb{R} | f \in (u(z) - z_0)^{-m}\mathbb{C}[u(z) - z_0]$. Thereby we define the number $m(u, f)$ which represent the minimal power of the function on the chosen basis.

In fourth section, we apply this formula on the complex exponential basis and we illustrate with the example $\cos^2(x)$ and $(1 + x)^a$.

Finally, in the fifth section, we apply the shifted Arm formula (this which is in the third section) on the periodic complex exponential basis and we identify the coefficients with the Fourier coefficients.

1 The Arm Formula

First we introduce the generalization to each basis $u(z)$ of the well known Taylor formula which is written in the basis $u(z) = z$ for each basis of the space $\mathbb{C}[u(z)-z_0]$ =span$\{1, (u(z)-z_0), (u(z)-z_0)^2, ...\}$

Theorem 1. *$\forall u(z) \in \mathcal{C}(\mathbb{C})$ if $\exists z \in \mathbb{C}$ such that $u(z) = z_0 \in \mathbb{C}$ then $\forall f(z) \in \mathbb{C}[u(z) - z_0]$*

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k f(z) \right] \big(u(z) - z_0\big)^k \tag{1.1}$$

Proof :

It's enough to show this formula on the basis $\left\{(u(z) - z_0)^p\right\}_{p \in \mathbb{N}}$.

If $k < p$:

$$\begin{aligned}
\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p &= \frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \partial^k_{u(z)} \big(u(z) - z_0\big)^p \\
&= \frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \frac{p!}{(p-k)!} \big(u(z) - z_0\big)^{p-k} \\
\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p &= 0 \qquad (1.2)
\end{aligned}$$

If $k > p$:

$$\begin{aligned}
\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p &= \frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial}{\partial u(z)} \right)^k \big(u(z) - z_0\big)^p \\
&= \frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \partial^{k-p}_{u(z)} p! \\
\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p &= 0 \qquad (1.3)
\end{aligned}$$

If $k = p$:

$$\begin{aligned}
\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p &= \lim_{z \to u^{-1}(z_0)} \frac{p!}{k!} \\
\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p &= 1 \qquad (1.4)
\end{aligned}$$

So we can see that :

$$\frac{1}{k!} \lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k \big(u(z) - z_0\big)^p = \delta_{k,p} \tag{1.5}$$

♦

Remark 1. *$f(z) \in \mathbb{C}[u(z)]$ means that $f(z)$ is a linear combination of elements of the basis $1, u(z), u(z)^2, \ldots$. So if you have a function which does not belong to this space, its Arm developpment will not be right. To give an example, $f(x) = \frac{1}{x} \notin \mathbb{C}[x]$ but $f(x) = \frac{1}{1-x} \in \mathbb{C}[x]$.*

Remark 2. *It's undertood that the operator $(\frac{\partial z}{\partial u}\frac{\partial}{\partial z})^k f(z)$ is defined by the recurrence relation :*

$$\left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^k f(z) = \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^{k-1} f(z)\right] \tag{1.6}$$

with the initialization

$$\left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^0 f(z) = f(z) \tag{1.7}$$

Remark 3. *The relation (1.1) means that the inner product of the space $\mathbb{C}[u(z) - z_0]$ is given by :*

$$< f, \big(u(z) - z_0\big)^k >= \frac{1}{k!} \left[\lim_{z \to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^k f(z) \right] \tag{1.8}$$

Remark 4. *It's well-know that the only function solution of first order differential equations is the exponential :*

$$\frac{\partial z}{\partial u}\frac{\partial}{\partial z} e^{u(z)} = e^{u(z)} \tag{1.9}$$

2 Examples Of The Arm Formula

In this section, if we don't say anything, we take the formula (1.1) with $z_0 = 0$ and $z = x \in \mathbb{R}$. We give examples for $u(x) = \frac{1}{x}, u(x) = \sqrt{x}$ and $u(x) = \cos(x)$ but it is the same for all $u(x)$.

- The Arm developpment on the inverse basis $u(x) = \frac{1}{x}$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to \infty} \left(- x^2 \frac{\partial}{\partial x}\right)^k f(x) \right] x^{-k} \tag{2.1}$$

For example, we can deduce from (2.1) that :

$$\frac{1}{1-x} = \lim_{x\to\infty} \frac{1}{1-x} + x \lim_{x\to\infty} -\frac{x^2}{(1-x)^2} + x^2 \frac{1}{2!} \lim_{x\to\infty} x^2 \frac{2x(1-x)^2 + 2x^2(1-x)}{(1-x)^4} + \ldots \tag{2.2}$$

which gives :

$$\frac{1}{1-x} = -\sum_{k=1}^{\infty} x^{-k} \tag{2.3}$$

Now we give usual inverses (on the basis $u(x) = \frac{1}{x}$) Arm developpments :

- The Arm developpment of $\frac{1}{1+x}$ on the inverse basis $u(x) = \frac{1}{x}$ is given by

$$\frac{1}{1+x} \quad = \quad -\sum_{k=1}^{\infty} \ (-1)^k \quad x^{-k} \tag{2.4}$$

- The Arm developpment of $(1+x)^a$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 1$ is given by

$$(1+x)^a \quad = \quad 2^a - 2^{a-1}a(x^{-1}-1) + 2^{a-3}a(a+3)(x^{-1}-1)^2 + ... \tag{2.5}$$

- The Arm developpment of e^x on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 1$ is given by

$$e^x \quad = \quad e - e\Big(x^{-1}-1\Big) + \frac{3e}{2}\Big(x^{-1}-1\Big)^2 - \frac{13e}{2}\Big(x^{-1}-1\Big)^3 + ... \tag{2.6}$$

- The Arm developpment of $\ln(1+\frac{1}{x})$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 0$ is given by

$$\ln\Big(1+\frac{1}{x}\Big) \quad = \quad x^{-1} - \frac{x^{-2}}{2} + \frac{x^{-3}}{3}... \tag{2.7}$$

- The Arm developpment of $\ln(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 1$ is given by

$$\ln\Big(x\Big) \quad = \quad -\Big(x^{-1}-1\Big) + \frac{1}{2}\Big(x^{-1}-1\Big)^2 - \frac{1}{3}\Big(x^{-1}-1\Big)^3 + ... \tag{2.8}$$

- The Arm developpment of $e^{\frac{1}{x}}$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 0$ is given by

$$e^{\frac{1}{x}} \quad = \quad 1 + x^{-1} + \frac{x^{-2}}{2} + ... \tag{2.9}$$

- The Arm developpment of $\sin(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = \frac{2}{\pi}$ is given by

$$\sin\Big(x\Big) \quad = \quad 1 - \frac{\pi^4}{32}\Big(x^{-1} - \frac{2}{\pi}\Big)^2 + \frac{\pi^5}{32}\Big(x^{-1} - \frac{2}{\pi}\Big)^3 + ... \tag{2.10}$$

- The Arm developpment of $\cos(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = \frac{1}{\pi}$ is given by

$$\cos(x) \quad = \quad -1 - \frac{\pi^4}{2}\Big(x^{-1} - \frac{2}{\pi}\Big)^2 - \frac{\pi^6(\pi^2-36)}{24}\Big(x^{-1} - \frac{1}{\pi}\Big)^4 + ... \tag{2.11}$$

- The Arm developpment of $\tan(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = \frac{4}{\pi}$ is given by

$$\tan(x) \quad = \quad 1 - \frac{\pi^2}{8}\Big(x^{-1} - \frac{4}{\pi}\Big) + \frac{\pi^3(4+\pi)}{128}\Big(x^{-1} - \frac{4}{\pi}\Big)^2 + ... \tag{2.12}$$

- The Arm developpment of $\sinh(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 1$ is given by

$$\sinh\left(x\right) = \sinh\left(1\right) - \cosh\left(1\right)\left(x^{-1} - 1\right) + \frac{(2\cosh(1) + \sinh(1))}{2}\left(x^{-1} - 1\right)^2 + ... \quad (2.13)$$

- The Arm developpment of $\cosh(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 1$ is given by

$$\cosh\left(x\right) = \cosh\left(1\right) - \sinh\left(1\right)\left(x^{-1} - 1\right) + \frac{(2\sinh(1) + \cosh(1))}{2}\left(x^{-1} - 1\right)^2 + ... \quad (2.14)$$

- The Arm developpment of $\tanh(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 1$ is given by

$$\tanh\left(x\right) = \tanh\left(1\right) - \frac{1}{\cosh(1)^2}\left(x^{-1} - 1\right) + \frac{8e^2}{(1+e^2)^3}\left(x^{-1} - 1\right)^2 + ... \quad (2.15)$$

- The Arm developpment of $\arccos(z) + i\ln(z)$ on the inverse basis $u(z) = \frac{1}{z}$ at $z_0 = 2$ is given by

$$\arccos\left(z\right) + i\ln(z) = -i\ln 2 + \frac{i}{4}z^{-2} + \frac{3i}{32}z^{-4} + \frac{5i}{96}z^{-6} + \frac{35i}{1024}z^{-8} + ... \quad (2.16)$$

which gives

$$\arccos\left(z\right) = -i\ln(z) - i\ln 2 + \frac{i}{4}z^{-2} + \frac{3i}{32}z^{-4} + \frac{5i}{96}z^{-6} + \frac{35i}{1024}z^{-8} + ... \quad (2.17)$$

- The Arm developpment of $\arcsin(z) - i\ln(z)$ on the inverse basis $u(z) = \frac{1}{z}$ at $z_0 = 2$ is given by

$$\arcsin\left(z\right) - i\ln(z) = \frac{\pi + 2i\ln 2}{6} - \frac{i}{4}z^{-2} - \frac{3i}{32}z^{-4} - \frac{5i}{96}z^{-6} - \frac{35i}{1024}z^{-8} + ... \quad (2.18)$$

which gives

$$\arcsin\left(z\right) = i\ln(z) + \frac{\pi + 2i\ln 2}{6} - \frac{i}{4}z^{-2} - \frac{3i}{32}z^{-4} - \frac{5i}{96}z^{-6} - \frac{35i}{1024}z^{-8} + ... \quad (2.19)$$

- The Arm developpment of $\arctan(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 0$ is given by

$$\arctan(x) = \frac{\pi}{2} - x^{-1} + \frac{x^{-3}}{3} - \frac{x^{-5}}{5} + \frac{x^{-7}}{7} - \frac{x^{-9}}{9} + ... \quad (2.20)$$

- The Arm developpment of $\operatorname{arcosh}(x) - \ln(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 2$ is given by

$$\operatorname{arcosh}\left(x\right) - \ln(x) = \ln 2 - \frac{1}{4}x^{-2} - \frac{3}{32}x^{-4} - \frac{5}{96}x^{-6} - \frac{35}{1024}x^{-8} + ... \quad (2.21)$$

which gives

$$\operatorname{arcosh}\left(x\right) = \ln(x) + \ln 2 - \frac{1}{4}x^{-2} - \frac{3}{32}x^{-4} - \frac{5}{96}x^{-6} - \frac{35}{1024}x^{-8} + ... \quad (2.22)$$

- The Arm developpment of $\operatorname{arsinh}(x) - \ln(x)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 2$ is given by

$$\operatorname{arsinh}\left(x\right) - \ln(x) \quad = \quad \ln 2 + \frac{1}{4}x^{-2} - \frac{3}{32}x^{-4} + \frac{5}{96}x^{-6} - \frac{35}{1024}x^{-8} + ... \tag{2.23}$$

which gives

$$\operatorname{arsinh}\left(x\right) \quad = \quad \ln(x) + \ln 2 + \frac{1}{4}x^{-2} - \frac{3}{32}x^{-4} + \frac{5}{96}x^{-6} - \frac{35}{1024}x^{-8} + ... \tag{2.24}$$

- The Arm developpment of $\operatorname{artanh}(z)$ on the inverse basis $u(x) = \frac{1}{x}$ at $z_0 = 2$ is given by

$$\operatorname{artanh}(z) \quad = \quad -\frac{i\pi}{2} + z^{-1} + \frac{z^{-3}}{3} + \frac{z^{-5}}{5} + \frac{z^{-7}}{7} + \frac{z^{-9}}{9} + ... \tag{2.25}$$

Next, we give Arm developpments on others basis :

- The Arm developpment on the square root basis $u(x) = \sqrt{x}$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0} \left(2\sqrt{x}\frac{\partial}{\partial x} \right)^k f(x) \right] \left(\sqrt{x} \right)^k \tag{2.26}$$

For example, we can deduce from (2.26) that :

$$\frac{\sqrt{x}}{\sqrt{x}-1} = \sum_{k=1} \frac{1}{k!} \left(\sqrt{x} \right)^k \lim_{x \to 0} (-1)^k \frac{k!}{(\sqrt{x}-1)^{k+1}} \tag{2.27}$$

which gives :

$$\frac{\sqrt{x}}{\sqrt{x}-1} \quad = \quad -\sum_{k=1}^{\infty} \left(\sqrt{x} \right)^k \tag{2.28}$$

- The Arm developpment on each power basis $u(x) = x^p$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0^{\operatorname{sgn}(p)}} \left(\frac{x^{1-p}}{p} \frac{\partial}{\partial x} \right)^k f(x) \right] \left(x^p \right)^k \tag{2.29}$$

- The Arm developpment on the real exponential basis $u(x) = e^x$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to -\infty} \left(e^{-x} \frac{\partial}{\partial x} \right)^k f(x) \right] \left(e^x \right)^k \tag{2.30}$$

- The Arm developpment on the complex exponential basis $u(z) = e^{iz}$ is given by

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{z \to i\infty} \left(-ie^{-iz} \frac{\partial}{\partial z} \right)^k f(z) \right] \left(e^{iz} \right)^k \tag{2.31}$$

- The Arm developpment on the logarithm basis $u(x) = \ln(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 1} \left(x \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\ln(x) \right)^k \tag{2.32}$$

- The Arm developpment on the cosinus basis $u(x) = \cos(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to \frac{\pi}{2}} \left(-\frac{1}{\sin(x)} \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\cos(x) \right)^k \tag{2.33}$$

For example, we can deduce from (2.33) that :

$$\begin{aligned} \cos 2x &= \lim_{x \to \frac{\pi}{2}} \cos(2x) + \cos^2(x) \frac{1}{2!} \lim_{x \to \frac{\pi}{2}} \frac{2\sin(2x) - 4\cos(2x)\tan(x)}{\sin^2(x)\tan(x)} \\ \cos(2x) &= -1 + 2\cos^2(x) \end{aligned} \tag{2.34}$$

- The Arm developpment on the sinus basis $u(x) = \sin(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0} \left(\frac{1}{\cos(x)} \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\sin(x) \right)^k \tag{2.35}$$

- The Arm developpment on the tangent basis $u(x) = \tan(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0} \left(\cos^2(x) \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\tan(x) \right)^k \tag{2.36}$$

- The Arm developpment on the hyperbolic cosinus basis $u(x) = \cosh(x)$ at the point $z_0 = 1$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0} \left(\frac{1}{\sinh(x)} \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\cosh(x) - 1 \right)^k \tag{2.37}$$

- The Arm developpment on the hyperbolic sinus basis $u(x) = \sinh(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0} \left(\frac{1}{\cosh(x)} \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\sinh(x) \right)^k \tag{2.38}$$

- The Arm developpment on the hyperbolic tangent basis $u(x) = \tanh(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{x \to 0} \left(-\cosh^2(x) \frac{\partial}{\partial x} \right)^k f(x) \right] \left(\tanh(x) \right)^k \tag{2.39}$$

3 The Shifted Arm Formula

If you have a function $\exists m \in \mathbb{R} | f \in (u(z)-z_0)^{-m}\mathbb{C}[u(z)-z_0]$, you can know it if the coefficients on the negative basis are zeros before the infinity.

Theorem 2. *$\forall u(z) \in \mathcal{C}(\mathbb{C})$ if $\exists z \in \mathbb{C}$ such that $u(z) = z_0 \in \mathbb{C}$ then*
$\forall f(z) \in (u(z)-z_0)^{-m}\mathbb{C}[u(z)-z_0]$

$$f(z) = \sum_{k=-m(u,f)}^{\infty} \frac{1}{(k+m(u,f))!} \left[\lim_{z\to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^{k+m(u,f)} (u(z)-z_0)^{m(u,f)} f(z) \right] \big(u(z)-z_0\big)^k \tag{3.1}$$

where the integer $m(u,f) \in \mathbb{N}$ is given by :

$$m(u,f) = \lim_{z\to u^{-1}(z_0)} -\frac{\ln(f(z))}{\ln(u(z)-z_0)} < \infty \tag{3.2}$$

<u>Proof :</u>
Let $f(z)$ has the decomposition

$$f(z) = \sum_{k=-m(u,f)}^{\infty} \alpha_k (u(z)-z_0)^k = \sum_{k=0}^{\infty} \alpha_{k-m(u,f)} (u(z)-z_0)^k (u(z)-z_0)^{-m(u,f)} \tag{3.3}$$

where $\alpha_k =< f, (u(z)-z_0)^k >$ defined in Remark 3. Pratically, we determine m in calculating

$$\begin{aligned}
\lim_{z\to u^{-1}(z_0)} -\frac{\ln(f(z))}{\ln(u(z)-z_0)} &= \lim_{z\to u^{-1}(z_0)} -\frac{\ln(\sum_{k=-m(u,f)}^{\infty} \alpha_k (u(z)-z_0)^k)}{\ln(u(z)-z_0)} \\
&= \lim_{z\to u^{-1}(z_0)} -\frac{\ln(\alpha_{-m(u,f)} (u(z)-z_0)^{-m(u,f)})}{\ln(u(z)-z_0)} \\
\lim_{z\to u^{-1}(z_0)} -\frac{\ln(f(z))}{\ln(u(z)-z_0)} &= m(u,f)
\end{aligned} \tag{3.4}$$

Inserting (3.3) in (1.1), we deduce

$$(u(z)-z_0)^{m(u,f)} f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\lim_{z\to u^{-1}(z_0)} \left(\frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)^{k} (u(z)-z_0)^{m(u,f)} f(z) \right] \big(u(z)-z_0\big)^k \tag{3.5}$$

from which we deduce (3.1) in changing $k' = k - m(u,f)$.

♦

Remark 5. $\big(u(z)-z_0\big)^{-m(u,f)} e^{(u(z)-z_0)} \in \ker((\mathrm{Ad}\big(u(z)-z_0\big)^{-m(u,f)})\partial_{u(z)} - 1)$ *means that*

$$\big(u(z)-z_0\big)^{-m(u,f)} \frac{\partial z}{\partial u}\frac{\partial}{\partial z} \big(u(z)-z_0\big)^{m(u,f)} \left[\big(u(z)-z_0\big)^{-m(u,f)} e^{\big(u(z)-z_0\big)} \right] = \left[\big(u(z)-z_0\big)^{-m(u,f)} e^{\big(u(z)-z_0\big)} \right] \tag{3.6}$$

Remark 6. *The logarithm of a complex variable is well defined :*

$$\ln(z) = \ln(|z|) + i\arg(z) \tag{3.7}$$

where $|z|$ is the modulus of z and arg(z) is its argument.

4 Examples

We consider the shifted Arm Formula on the complex exponential basis $u(z) = e^{iz}$:

$$f(z) = \sum_{k=-m(e^{iz},f)}^{\infty} \frac{1}{(m(e^{iz},f)+k)!} \left[\lim_{z\to i\infty} \left(-ie^{-iz}\frac{\partial}{\partial z}\right)^{m(e^{iz},f)+k} e^{im(e^{iz},f)z} f(z) \right] \left(e^{iz}\right)^k \tag{4.1}$$

with

$$m(e^{iz},f) = \lim_{z\to i\infty} -i\frac{\ln(f(z))}{z} \tag{4.2}$$

We try this formula on the quite simple example of $f(z) = \cos^2(z)$. First we have to calculate the minimal power $m(e^{iz}, \cos^2(z))$ of the polynomial function $\cos^2(z)$

$$m(e^{iz}, \cos^2(z)) = \lim_{z\to i\infty} -i\frac{\ln(\cos^2(z))}{z} = 2 \tag{4.3}$$

since $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$. Now, we can apply the formula (4.1) and we obtain :

$$\begin{aligned}
\cos^2 z &= e^{-2iz}\left(\lim_{z\to i\infty} e^{2iz}\cos^2 z\right) + \frac{1}{2!}\left(\lim_{z\to i\infty} 1+3\cos(2z)+3i\sin(2z)\right) \\
&\quad + \frac{1}{4!}e^{2iz}\left(\lim_{z\to i\infty} 6\right) \\
\cos^2 z &= \frac{e^{-2iz}+2+e^{2iz}}{4}
\end{aligned} \tag{4.4}$$

I give the example of $f(z) = \cos^2 z$ because it is the simplest non trivial function but of course the theory works for each functions $f(z) \in e^{-2iz}\mathbb{C}[e^{iz}]$.

Next, we consider the shifted Arm Formula on the inverse basis $u(x) = x^{-1}$:

$$f(x) = \sum_{k=-m(x^{-1},f)}^{\infty} \frac{1}{(m(x^{-1},f)+k)!} \left[\lim_{x\to\infty} \left(-x^2\frac{\partial}{\partial x}\right)^{m(x^{-1},f)+k} x^{-m(x^{-1},f)} f(x) \right] \left(x\right)^{-k} \tag{4.5}$$

with

$$m(x^{-1},f) = \lim_{x\to\infty} \frac{\ln(f(x))}{\ln(x)} \tag{4.6}$$

We try this formula on the example $f(x) = (1+x)^a$. First we have to calculate the minimal power $m(x^{-1}, (1+x)^a)$ of the function $(1+x)^a$

$$m(x^{-1}, (1+x)^a) = \lim_{x\to\infty} \frac{\ln((1+x)^a)}{\ln(x)} = a \tag{4.7}$$

Now, we can apply the formula (4.5) and we obtain :

$$\begin{aligned}
(1+x)^a &= x^a + ax^{a-1} + \frac{a(a-1)}{2!}x^{a-2} + \frac{a(a-1)(a-2)}{3!}x^{a-3} + ... \\
(1+x)^a &= \sum_{k=0}^{\infty} \binom{a}{k} \; x^{a-k}
\end{aligned} \tag{4.8}$$

Note that we can also find (4.8) in inserting $x' = x^{-1}$ in the traditional Taylor developpment of $(1+x)^a$. We see that $(1+x)^a \in x^a\mathbb{C}[x^{-1}]$.

The equation (4.8) for $a = -\frac{1}{2}$ and $-x^2$ instead of x :

$$\frac{1}{\sqrt{x^2-1}} \quad = \quad x^{-1} + \frac{1}{2}x^{-3} + \frac{3}{8}x^{-5} + \frac{5}{16}x^{-7} + \frac{35}{129}x^{-9} + ...$$

Integrating (4.9), we obtain (2.24) because $(x^2-1)^{-\frac{1}{2}}$ is the derivative of $\mathrm{arsinh}(x)$. In doing the same for arcosh, artanh, arcsin and arccos, we obtain (2.22), (2.25), (2.19) and (2.17) .

Next, we consider the shifted Arm Formula on the natural basis $u(x) = x$:

$$f(x) = \sum_{k=-m(x,f)}^{\infty} \quad \frac{1}{(m(x,f)+k)!} \quad \left[\lim_{x\to 0} \left(\frac{\partial}{\partial x}\right)^{m(x,f)+k} x^{m(x,f)} f(x) \quad \right] \quad \Big(x\Big)^k \tag{4.9}$$

with

$$m(x,f) = \lim_{x\to 0} -\frac{\ln(f(x))}{\ln(x)} \tag{4.10}$$

We try this formula on the example $f(x) = \frac{x^2+1}{x^4(x^2+x+1)}$. First we have to calculate the minimal power $m(x, \frac{x^2+1}{x^4(x^2+x+1)})$ of the function $(1+x)^a$

$$m(x, \frac{x^2+1}{x^4(x^2+x+1)}) = \lim_{x\to 0} -\frac{\ln((\frac{x^2+1}{x^4(x^2+x+1)})}{\ln(x)} = 4 \tag{4.11}$$

Now, we can apply the formula (4.9) and we obtain :

$$\frac{x^2+1}{x^4(x^2+x+1)} \quad = \quad x^{-4} - x^{-3} + x^{-2} - 1 + x - x^3 + x^4 - x^6 + x^7 - x^9 + x^{10} + ... \tag{4.12}$$

Now that I convince you that my formula works, I can do the link with the Fourier series and identifying the Fourier coefficients with the Arm corfficients.

5 Link With Fourier Series

We now consider the shifted Arm formula (3.1) for a basis $u(z) = e^{\frac{2i\pi z}{T}}$ where T is a period :

$$f(z) = \sum_{k=-m_F}^{\infty} \quad \frac{1}{(m_F+k)!} \quad \left[\lim_{z\to i\infty} \left(\frac{-iT}{2\pi} e^{\frac{-2i\pi z}{T}} \frac{\partial}{\partial z}\right)^{m_F+k} e^{im_F z} f(z) \quad \right] \quad \left(e^{\frac{2i\pi z}{T}}\right)^k \tag{5.1}$$

with

$$m_F = \lim_{z\to i\infty} -\frac{\ln(f(z))}{\frac{2i\pi z}{T}} \tag{5.2}$$

and

$$m_F = m(e^{\frac{2i\pi z}{T}}, f) \tag{5.3}$$

So we can identify

Proposition 1. *The inner product of Fourier and the inner product of Arm on the complex exponential basis are equal :*

$$\begin{aligned} < f, e^{\frac{2i\pi z}{T}} >_F &= < f, e^{\frac{2i\pi z}{T}} >_A \\ \frac{1}{T}\int_0^T e^{\frac{-2ik\pi z}{T}} f(z)\mathrm{d}z &= \frac{1}{(m_F+k)!} \left[\lim_{z\to i\infty} \left(\frac{-iT}{2\pi} e^{\frac{-2i\pi z}{T}} \frac{\partial}{\partial z}\right)^{m_F+k} e^{im_F z} f(z) \right] \end{aligned} \tag{5.4}$$

<u>**Proof :**</u>

By unicity of the decomposition of a function $f(z) \in e^{\frac{-2i\pi z m_F}{T}} \mathbb{C}[e^{\frac{2i\pi z}{T}}]$.

♦

Discussion

The Fourier series are decompositions on periodic functions basis (the complex exponential basis), we then created the Fourier theory to extend this decompositions on non-periodic functions.

Here there is no need to do the same thing because, for a given function, you have to find a right basis on your function where it can be decompose. The difficulty here is to find this right basis. First, I tried to do the same as we do for the Fourier series i.e. I introduced a basis $\{(v(z)-z_0)^{\frac{k}{n}}\}_{k\in\mathbb{N}}$ such that $(v(z)-z_0)^{\frac{1}{n}} = u(z) - z_0$ and I take the limite when $n \to \infty$. This is quite the same method as taking $T \to \infty$ in (5.1). I dreamt to obtain an Arm formula on 'continue' basis and not discret basis i.e. z^a with $a \in \mathbb{R} \neq \mathbb{N}$. As a consequence I arrived to the expression :

$$f(z) = \int_{-m(v,f)}^{\infty} \mathrm{d}k \left(\lim_{n\to\infty} \frac{1}{(nk+nm(v,f))!} \left[\lim_{z\to v^{-1}(z_0)} \partial^{nk}_{v(z)^{\frac{1}{n}}} (v(z)-z_0)^{m(v,f)} f(z) \right] \right) \Big(v(z) - z_0 \Big)^k$$

and I wanted to identify the coefficient with the Fourier transformed when $v(z) = e^{iz}$ and $z_0 = 0$. After all, I mention this formula as a historical error because it is useless and we can't use it.

The mathematical developpment done for the Fourier theory is useless here because we developpe the Fourier transform in order to extend the Fourier decomposition to nonperiodic functions. But here there is no need to do it because you just have to find an appropriate basis for your function and then you can decompose it.