The p-Arm Theory

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Keywords : Series, Exponential
MSC : 40-00

Abstract
We introduce the p-Arm theory which give rise to a new mathematical object that we call the "p-exponential" which is invariant under p derivation. We calculate its derivate and we use this new function to solve differential equations. Next, we define its real and imaginary part which are the p-cosinus and the p-sinus respectively.
Introduction

The Arm theory [1] gives a development on any p-th power function basis in changing of variable in the Arm formula. But for functions in \( C[(u(z) - z_0)^p], p \in \mathbb{N}^* \) there is an other way (the p-Arm formula) to make this development: instead of changing the variable at the p-th powers, you can also derivate p times which will finally give the same result. This is the main idea behind the p-Arm theory.

The exponential function is the function which leaves invariant the operator in the Taylor formula i.e.:
\[
\frac{\partial e^x}{\partial x} = e^x
\]  
(0.1)

So in constructing the p-Arm theory, we see that we need a "p-exponential" \( e_p^x \) function which leaves the operator of the p-Arm formula invariant:
\[
\frac{\partial^p e_p^x}{\partial x^p} = e_p^x; \quad \frac{\partial^k e_p^x}{\partial x^k} \neq e_p^x
\]  
(0.2)

for \( 1 \leq k < p \). The answer to the question (0.2) is the definition of the p-exponential as follow:
\[
e_p^x = \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!}
\]  
(0.3)

The p-Arm formula is not so much interesting itself because we already have the development by the Arm-theory, but this formula give rise to the p-exponential which is very interesting to study.

In studying the derivate of the p-exponential, we see that this operator acts like a shift operator on the p-exponential and we need a generalization of the "p-exponential" to also include its derivate. This generalized exponential function is:
\[
e_{p,\mu}^x = \sum_{k=0}^{\infty} \frac{x^{pk+\mu}}{(pk+\mu)!}
\]  
(0.4)

for \( p, \mu \in \mathbb{N}^* \). I know that there is already a generalized exponential function in the theory of the fractional calculus (see [2]) which is given by
\[
E^y_{\mu} \equiv \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)}
\]  
(0.5)

but which one I introduce here is more generalized because (0.4) has a multiplication and a shift whereas (0.5) has only a shift.
In the first section, we give the equivalent of the Arm formula for the p-Arm theory which we naturally call the p-Arm formula for function in $\mathbb{C}[(u(z) - z_0)^p]$.

In the second section, we give the equivalent shifted Arm formula for the p-Arm theory which we call the shifted p-Arm formula.

In the third section, we give the definition of the generalized exponential function. Next, we draw the six first real p-exponentials which is a beautiful graph. In effect, we explain why the p-th derivate of the p-exponential is itself. In this case, we calculate the derivate of the p-exponential. Thereby, we give the relation between the p-exponential and the traditional exponential. This is why we use this result to show that every function solving that its p-th derivate is itself can be expressed as a linear combination of p-exponential and we give the example of $p = 2$. Then defining the complex p-exponential, we give its real part called the p-cosinus and we draw the six first p-cosinus. Furthermore, we define the p-sinus which is the imaginary part of the complex p-exponential and we draw the six first of it. Finally, we define the p-tangent and we draw the six first p-tangent.
1 The p-Arm Formula

First we introduce the generalization to each basis $u(z)$ of the well known Taylor formula which is written in the basis $u(z) = z$ for each basis of the space $\mathbb{C}[(u(z) - z_0)^p] = \text{span}\{1, (u(z) - z_0)^p, (u(z) - z_0)^{2p}, \ldots\}$

**Theorem 1.** $\forall u(z) \in \mathbb{C}(\mathbb{C})$ if $\exists z \in \mathbb{C}$ such that $u(z) = z_0 \in \mathbb{C}$ then $\forall f(z) \in \mathbb{C}[(u(z) - z_0)^p]$

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[ \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} f(z) \right] (u(z) - z_0)^{pk} \quad (1.6)$$

**Proof:**

It’s enough to show this formula on the basis $\{(u(z) - z_0)^p\}_{r \in \mathbb{N}}$.

If $k < r$:

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \partial_{u(z)}^{pk} (u(z) - z_0)^{pr} = \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} (pr)! (u(z) - z_0)^{p(r-k)}$$

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = 0 \quad (1.7)$$

If $k > r$:

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial}{\partial u(z)} \right)^{pk} (u(z) - z_0)^{pr} = \frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \partial_{u(z)}^{p(k-r)} (pr)!$$

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = 0 \quad (1.8)$$

If $k = r$:

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = \lim_{z \to u^{-1}(z_0)} (pr)! \frac{(pk)!}{(pk)!}$$

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = 1 \quad (1.9)$$

So we can see that:

$$\frac{1}{(pk)!} \lim_{z \to u^{-1}(z_0)} \left( \frac{\partial^k}{\partial u \partial z} \right)^{pk} (u(z) - z_0)^{pr} = \delta_{k,r} \quad (1.10)$$

$\diamondsuit$
2 The Shifted p-Arm Formula

If you have a function \( f \in \mathbb{C}[(u(z) - z_0)^{-p}] \oplus \mathbb{C}[(u(z) - z_0)^p] \), you can know it if the coefficients on the negative basis are zeros before the infinity.

**Theorem 2.** \( \forall u(z) \in \mathbb{C}(z) \) if \( \exists z \in \mathbb{C} \) such that \( u(z) = z_0 \in \mathbb{C} \) then
\[
f(z) = \sum_{k=-m_p(u,f)}^{\infty} \frac{1}{(p(k + m_p(u,f)))!} \left[ \ln(f(z)) - \frac{\ln(f(z))}{p \ln(u(z) - z_0)} \right] z^k \]
where the integer \( m_p(u,f) \in \mathbb{N} \) is given by :
\[
m_p(u,f) = \lim_{z \to u_i^{-1}(z_0)} \frac{\ln(f(z))}{p \ln(u(z) - z_0)} < \infty
\]

**Proof :**
Let \( f(z) \) has the decomposition
\[
f(z) = \sum_{k=-m_p(u,f)}^{\infty} \alpha_p(u(z) - z_0)^{pk} = \sum_{k=0}^{\infty} \alpha_{p(k-m_p(u,f))}(u(z) - z_0)^{pk}(u(z) - z_0)^{-pm_p(u,f)}
\]
where \( \alpha_k = \langle f, (u(z) - z_0)^k \rangle \). Pratically, we determine \( m_p \) in calculating
\[
\lim_{z \to u_i^{-1}(z_0)} \frac{\ln(f(z))}{p \ln(u(z) - z_0)} = \lim_{z \to u_i^{-1}(z_0)} \frac{\ln(\sum_{k=-m(u,f)}^{\infty} \alpha_k(u(z) - z_0)^k)}{p \ln(u(z) - z_0)} = \lim_{z \to u_i^{-1}(z_0)} \frac{\ln(\alpha_{-m(u,f)}(u(z) - z_0)^{-m(u,f)})}{p \ln(u(z) - z_0)} = m(u,f)
\]
Inserting (2.13) in (1.6), we deduce
\[
(u(z)-z_0)^{pm_p(u,f)} f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[ \lim_{z \to u_i^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^k (u(z)-z_0)^{pm_p(u,f)} f(z) \right] (u(z)-z_0)^{pk}
\]
from which we deduce (2.11) in changing \( k' = k - m_p(u,f) \).

**Remark 1.** If you consider the shifted p-Arm formula (2.11) for \( p = 2, u(z) = e^{iz} \) and \( z_0 = 0 \), you will check that :
\[
\cos^2(z) = \frac{e^{2iz} + 2 + e^{-2iz}}{4}
\]
with \( m_2(e^{iz}, \cos^2) = 1 \).
The shifted p-Arm formula gives rise to a new mathematical function which make one the limit in the formula (2.11).

3 The p-exponential

**Definition 1.** We define the generalised exponential function:

\[
e_{p,\mu}^x = \sum_{k=0}^{\infty} \frac{x^{kp+\mu}}{(kp+\mu)!}
\]

for \( p, \mu \in \mathbb{N}^* \).

In the rest of this paper, we will call \( e_{p,0}^x = e^x_p \) the "p-exponential".

Now because we want see what are these new function, we draw the 6 first real p-exponentials:

![Figure 1 - The six first p-exponentials](image)

We now explain why is this function interesting

**Proposition 1.** The p-exponential is a function such that

\[
\frac{\partial^p e_p^x}{\partial x^p} = e_p^x \quad \text{and} \quad \frac{\partial^l e_p^x}{\partial x^l} \neq e_p^x
\]

for each \( 1 \leq l < p \).
Proof:

\[
\frac{\partial e_p^x}{\partial x^p} = \frac{\partial}{\partial x^p} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!}
\]

\[
= \sum_{k=1}^{\infty} \frac{(pk)!}{(pk - p)!} \frac{x^{pk}}{(pk)!}
\]

\[
= \sum_{k=1}^{\infty} \frac{x^{pk-p}}{(pk-p)!}
\]

\[
= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!}
\]

\[
\frac{\partial e_p^x}{\partial x^p} = e_p^x
\]  \hspace{1cm} (3.19)

The second part of (3.18) is trivial.

Now, we calculate the derivative of the p-exponential

**Proposition 2.** The derivate of the p-exponential is given by:

\[
\frac{\partial e_p^x}{\partial x} = e_{p,p-1}^x
\]  \hspace{1cm} (3.20)

where \( p \in \mathbb{N}^* \).

**Proof:**

\[
\frac{\partial e_p^x}{\partial x} = \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!}
\]

\[
= \sum_{k=1}^{\infty} \frac{(pk)!}{(pk-1)!} \frac{x^{pk-1}}{(pk)!}
\]

\[
= \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!}
\]

\[
= \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!}
\]

\[
\frac{\partial e_p^x}{\partial x} = e_{p,p-1}^x
\]  \hspace{1cm} (3.21)

\[\dag\]
Remark 2. Of course we have
\[
\frac{\partial e^u(x)}{\partial x} = \frac{\partial u}{\partial x} e^{u(x)} e_p^{u(x)} \quad (3.22)
\]

Remark 3. We see that because of (3.20), we have:
\[
\frac{\partial^k e^x_p}{\partial x^k} = e^{x-p-k} \quad (3.23)
\]
for \(1 \leq k \leq p\). So the derivation acts like a shift operator on the p-exponential.

Now we show an interesting relation which links the p-exponential with the traditional exponential.

Proposition 3. The link between the p-exponential and the usual exponential is given by:
\[
\left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e^x_p = e^x \quad (3.24)
\]
or equivalently:
\[
\sum_{\mu=0}^{p-1} e_{p,\mu} e^x_p = e^x \quad (3.25)
\]

Proof:
\[
\left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e^x_p = e^x_p + \frac{\partial}{\partial x} e^x_p + \ldots + \frac{\partial^{p-1}}{\partial x^{p-1}} e^x_p \\
= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \ldots + \frac{\partial^{p-1}}{\partial x^{p-1}} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} + \ldots + \sum_{k=1}^{\infty} \frac{x^{pk-p+1}}{(pk-p+1)!} \\
= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} + \ldots + \sum_{k=0}^{\infty} \frac{x^{pk+1}}{(pk+1)!} \\
= e^x_p + e_{p,p-1} e^x_p + \ldots + e_{p,1} e^x_p
\]
\[
\left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e^x_p = e^x \quad (3.26)
\]

Now we introduced the p-exponential, we can use it to solve some differential equations. In fact, this is why I created it, the exponential solves the limit of the first order differential equation in the traditional Taylor formula whereas the p-exponential solves the limit of the pth order differential equation in (1.6).
Proposition 4. Let the differential equation
\[
\frac{\partial^p u(x)}{\partial x^p} = u(x) \tag{3.27}
\]
exists \(\alpha_1, \ldots, \alpha_p\) such that the solution of (3.27) can be expressed as:
\[
u(x) = \sum_{k=1}^{p} \alpha_k \ e_p \omega_p^k x \tag{3.28}
\]
where \(\omega_p = e^{\frac{2\pi}{p}}\) is the \(p\)-th root of unity.

Proof:
\[
\frac{\partial^p u(x)}{\partial x^p} = \sum_{k=1}^{p} \alpha_k \ \frac{\partial^p \omega_p^k x}{\partial x^p} = \sum_{k=1}^{p} \alpha_k \left( \frac{\partial (\omega_p^k x)}{\partial x} \frac{\partial}{\partial (\omega_p^k x)} \right)^p \ e_p \omega_p^k x \\
= \sum_{k=1}^{p} \alpha_k \omega_p^k \ e_p \omega_p^k x
\]
\[
\frac{\partial^p u(x)}{\partial x^p} = u(x) \tag{3.29}
\]

Example:
As an example of (3.27), we solve the well-known case:
\[
\frac{\partial^2 u(x)}{\partial x^2} = u(x) \tag{3.30}
\]
The formula (3.28) gives the solution:
\[
u(x) = \alpha_1 \ e^x + \alpha_2 \ e^{-x} \\
u(x) = \alpha_1 \cosh(x) + \alpha_2 \cosh(-x) \tag{3.31}
\]
where \(\alpha_1, \alpha_2 \in \mathbb{C}\) depend on the initial conditions.
Now we define the p-cosinus and p-sinus functions

**Definition 2.** The p-cosinus is the real part of the complex exponential given by

\[
\cos_p(x) = \frac{e_p^{ix} + e_p^{-ix}}{2}
\]

(3.32)

We draw the 6 first p-cosinus

![Figure 2 – The six first p-cosinus](image)

**Definition 3.** The p-sinus is the imaginary part of the complex exponential given by

\[
\sin_p(x) = \frac{e_p^{ix} - e_p^{-ix}}{2i}
\]

(3.33)

We draw the 6 first p-sinus

![Figure 3 – The six first p-sinus](image)
Definition 4. The $p$-tangent is given by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)}$$

(3.34)

We draw the 6 first $p$-tangent

Figure 4 – The six first $p$-tangent
Discussion

Even if the limit of the sum of two elements seems to be
\[
\lim_{x+y \to \infty} e^{x+y} = \frac{1}{p} e^x e^y \tag{3.35}
\]
on the graph for \( p \geq 2 \), I didn’t find a simple relation between the sum of arguments and the product of exponentials. In a same way, we don’t have an equivalent of the Moivre formula which links the \( n \)-th power of the exponential with the multiplication with \( n \) of the argument. However this relation seems to exist on the graph if we consider it in the infinity limit :
\[
\lim_{x \to \infty} e^{nx} = \frac{1}{p} (e^x)^n \tag{3.36}
\]
for \( p \geq 2 \)

In addition I also search for the value of the module of the \( p \)-exponential but it seems to not have a fixed valued on the graph. So on the graph, it seems to be :
\[
\lim_{x \to \infty} \cos^2_p(x) + \sin^2_p(x) = \infty \tag{3.37}
\]
for \( p \geq 3 \). There is an exception for \( p = 2 \) because \( e_2 = \cosh \) and we have that :
\[
|e_2^{ix}| = \cos(x) \tag{3.38}
\]

For now, I didn’t find yet the inverse function of the \( p \)-exponential or of the generalized exponential function. I tried finding an expression for the derivate of the ”\( p \)-logarithm” :
\[
\left. \frac{\partial \ln_p(x)}{\partial x} \right|_{x=e_p^x} = \frac{1}{e_p^{x} e_{p-1}} \tag{3.39}
\]
but we need a relation between the \( p \)-exponential \( e_p^x \) and its derivate \( e_p^{x} e_{p-1} \) other than the derivation relation itself.
Références