The Arm Lie Group Theory

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Abstract
We develop the Arm-Lie group theory which is a theory based on the exponential of a changing of matrix variable $u(X)$. We define a corresponding $u$-adjoint action, the corresponding commutation relations in the Arm-Lie algebra and the $u$-Jacobi identity. Throught the exponentiation, Arm-Lie algebras become Arm-Lie groups. We give the example of $\sqrt{\mathfrak{so}(2)}$ and $\sqrt{\mathfrak{su}(2)}$. 
Introduction

The Arm theory [1] gives the generalized Taylor formula in any basis. It gives rise to exponentials of any changing of variable $u(x)$. While the Lie group theory has been built on the classical exponential, I wondered myself why not building a new Lie group theory based on exponential of changing of variable, i.e. on exponential:

$$e^{u(X)}$$ (0.1)

where $u(X)$ is any changing of variable.

Besides, we can build a new adjoint action with this exponential in a new basis. This is the 'u-adjoint action' given in proposition 1:

$$\text{ad}_u A.X = [A, u(X)]$$ (0.2)

The traditional Lie algebra satisfied some commutation relations between their generators so I searched the corresponding commutation relations for this new structure. In fact the generators of the Arm-Lie algebras $\{X_1, \ldots, X_n, u(X_1), \ldots, u(X_n)\}$, if $n$ is the dimension, satisfy the following conditions:

$$[X_i, u(X_j)] = \lambda_{ij}^k u(X_k)$$
$$[X_i, X_j] = \nu_{ij}^k u(X_k)$$ (0.3)

with $\lambda_{ij}^k$ and $\nu_{ij}^k$ the corresponding u-structure constant and structure constant respectively. In addition, there is a corresponding u-Jacobi identity in Arm-Lie algebras which is given by:

$$[u(X), [Y,Z]] + [u(Y), [Z,X]] + [u(Z), [X,Y]] = 0$$ (0.4)

In the same way, Lie algebras become Lie groups through exponentiation, we can take the exponential of each linear combinations which gives a group that we call a Arm-Lie group. This is why we check in the third section that the commutator of commutators of elements in the Arm-Lie algebras are still in the Arm-Lie algebra.

However, the big default of my construction is that I only find one example of the Arm-Lie algebra, but what a beautiful example: $u^{-1}(\mathfrak{so}(2))$ and $u^{-1}(\mathfrak{su}(2))$ with $\forall p \in \mathbb{C}, u(x) = X^p$. I think it is because the generators of $\mathfrak{su}(2)$ and $\mathfrak{so}(2)$ are the exponential of something (i.e. their logarithm exist) but I am not sure yet. Nevertheless I hope there is other examples of Arm-Lie algebras. Moreover, the exponential of those two Arm-Lie algebras give two new groups which are Arm-Lie groups that we call $rSO(2)$ and $rSU(2)$ given by

$$rSO(2, \mathbb{C}) = \left\{ M \in \mathbb{M}_2(\mathbb{C}) \mid {}^tMM = r\text{Id}_2 \ ; \ r \in \mathbb{C} \right\}$$ (0.5)

and

$$rSU(2, \mathbb{C}) = \left\{ M \in \mathbb{M}_2(\mathbb{C}) \mid MM^\dagger = r\text{Id}_2 \ ; \ r \in \mathbb{C} \right\}$$ (0.6)
In the first section, we define the exponential in each basis of function $u(X)$ of matrices. Next in the second section, we give the commutation relations between generators of the Arm-Lie algebra and the $u$-Jacobi identity. After in the third section, we show that commutators of commutators of elements of the Arm-Lie algebras are still in the Arm-Lie algebra. This is the condition which assure that the exponential of linear combinations will be groups which we call Arm-Lie groups. Furthermore in the fourth section we give the example of the $p$-th root of $\mathfrak{so}(2)$ which is a trivial case because there is only two generators in the Arm-Lie algebra. Nonetheless it give rise to the new group $rSO(2)$ (0.5) and we give elements of this groups. Finally in the fifth section, we give the example of the $p$-th root of $\mathfrak{su}(2)$ which is not trivial because there is 6 generators in this Arm-Lie algebra. We give generating elements of its corresponding Arm-Lie group $rSU(2)$ (0.6).
1 The U-Exponential Map

Definition 1. We call the 'u-exponential' the function

\[ e^{u(X)t} = \sum_{k=0}^{\infty} \frac{(u(X)t)^k}{k!} \] (1.7)

where \( u(X) \in C(M_l(C)), l \in \mathbb{N} \) is a function of matrices and \( X \) is a matrix.

Then we can compute the corresponding u-adjoint action

Proposition 1. The u-adjoint action corresponding to the u-exponential is given by

\[ \text{ad}_u A.X = [A, u(X)] \] (1.8)

where \([, ,]\) is the traditional Lie bracket.

Proof:

\[
\begin{align*}
\text{ad}_u A.X &= \lim_{t \to 0} \frac{d}{dt} \left[ e^{-u(X)t} A e^{u(X)t} \right] \\
&= \lim_{t \to 0} \frac{d}{dt} \left[ (1 - u(X)t + ...) A (1 + u(X)t + ...) \right] \\
&= \lim_{t \to 0} \frac{d}{dt} \left[ A + t(Au(X) - u(X)A) + .. \right] \\
&= Au(X) - u(X)A \\
\text{ad}_u A.X &= [A, u(X)] \quad (1.9)
\end{align*}
\]

We call \( g \) a Lie algebra.

2 The Arm-Lie Algebra

Definition 2. A Arm-Lie algebra is a collection \( \{X_1, ..., X_n, u(X_1), ..., u(X_n)\} \) such that \( \{u(X_1), ..., u(X_n)\} \) is a basis of the Lie algebra \( g \).

In the Arm-Lie algebra, we have the following relations

\[
\begin{align*}
[X_i, u(X_j)] &= \lambda_{ij}^k u(X_k) \\
[X_i, X_j] &= \nu_{ij}^k u(X_k) \quad (2.10)
\end{align*}
\]

where \( \nu \) and \( \lambda \) are the structure constant and the u-structure constant respectively.

We can also define the corresponding u-Jacobi identity

\[
[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0 \quad (2.11)
\]
3 The Arm-Lie Group

First we recall the Campbell-Hausdorff formula

$$e^X e^Y = e^{Z(X,Y)}$$

(3.12)

where

$$Z(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X - Y, [X,Y]]) - \frac{1}{24}[Y, [X, [X, Y]]] + ...$$

(3.13)

The idea of the Arm-Lie group is that if we take generators of the Arm-Lie algebra $X = X_i$ and $Y = X_j$ then each terms of $Z(X,Y)$ in (3.13) would be the commutators of elements of the Arm-Lie algebra:

$$[X_k, ..., [X_f, [X_p, X_l]]] = [X_k, ..., [X_f, \nu_{pl}^r u(X_l)]]$$

$$= \nu_{pj}^r [X_k, ..., \lambda_{jl}^r u(X_l)]$$

$$...$$

$$[X_k, ..., [X_f, [X_p, X_l]]] = \nu_{pj}^r \lambda_{km}^{sl} u(X_s) + ... + \lambda_{ab}^e \lambda_{de}^f u(X_f)$$

(3.14)

and the first term of $Z(X,Y)$ are given by $X_i + X_j$. Then we can conclude that the exponential of elements of an Arm-Lie algebra is a group which we naturally call Arm-Lie group.

In the moment I’m writing this article, the only Lie algebra which give Arm-Lie algebras are Lie algebra with generators which are the exponential of something (i.e. their logarithms exist). So we can start in studying the trivial one dimensional Arm-Lie algebra $\sqrt[4]{so(2)}$ and its corresponding Arm-Lie group.

4 $\sqrt[4]{so(2)}$

We consider the changing of variable:

$$u(X) = X^p$$

(4.15)

for $p \in \mathbb{C}$. It’s well known that the generator basis of $so(2)$ is given by the first Pauli matrix multiply by $\sqrt{-1}$:

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p$$

(4.16)

Because $i\sigma_1$ is a basis of $so(2)$, if we want to know the basis of $u^{-1}(so(2)) = \sqrt[4]{so(2)}$, we have to calculate the p-th root of $i\sigma_1$. Then you can check the validity of the relation:

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

(4.17)
So with (4.17), it is very easy to calculate the p-th root of $i\sigma_1$:

\[
\sqrt[i]{i\sigma_1} = \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)
\]

\[
= \cos\left(\frac{\pi}{2p}\right) \text{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

\[
X_1 = \sqrt[i]{i\sigma_1} = \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix}
\]

where $\text{Id}_2$ is of course the identity 2-dimensional matrix. This Arm-Lie algebra is trivial because there is only 2 generators $i\sigma_1 = (X_1)^p$ and $\sqrt[i]{i\sigma_1} = X_1$ satisfying the relations:

\[
[X_1, X_1^p] = 0
\]

\[
[X_1, X_1] = 0
\]

(4.18)

Of course (4.18) implies that the generator of the Arm-Lie algebra satisfy the u-Jacobi identity (2.11). Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie group:

\[
\exp(\sqrt[i]{so(2)}) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid e^{z_1 \sqrt[i]{i\sigma_1} + z_2 i\sigma_1} \right\}
\]

(4.19)

To have an idea of the elements of the group (4.19), we can explicit:

\[
\exp(z_1 X_1) = \exp\left(z_1 \cos\left(\frac{\pi}{2p}\right) \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix} \right)
\]

(4.20)

and the well-known element of $SO(2)$:

\[
\exp(z_2 X_1^p) = \begin{pmatrix} \cos\left(z_2\right) & -\sin\left(z_2\right) \\ \sin\left(z_2\right) & \cos\left(z_2\right) \end{pmatrix}
\]

(4.21)

Hence we can identify this Arm-Lie algebra to

\[
\exp(\sqrt[i]{so(2)}) \equiv rSO(2, \mathbb{C}) = \left\{ M \in M_n(\mathbb{C}) \mid \text{tr} MM = r \text{Id}_2 ; \ r \in \mathbb{C} \right\}
\]

(4.22)

for $\frac{1}{p} \neq 1[2]$ and just $SO(2)$ if $\frac{1}{p} = 1[2]$.

**Remark 1.** Of course the cosinus and the sinus of a complex arument is well defined

\[
\cos(z) = \cos(\text{Re}(z)) \cosh(\text{Im}(z)) - i \sin(\text{Re}(z)) \sinh(\text{Im}(z))
\]

\[
\sin(z) = \sin(\text{Re}(z)) \cosh(\text{Im}(z)) + i \cos(\text{Re}(z)) \sinh(\text{Im}(z))
\]

(4.23)

where $\text{Re}$ and $\text{Im}$ denote the real and imaginary parts respectively.
We still consider the changing of variable:

\[ u(X) = X^p \quad (5.24) \]

for \( p \in \mathbb{C}^* \). It's well known that the generator basis of \( \mathfrak{su}(2) \) is given by the Pauli matrices multiply by \( \sqrt{-1} \):

\[
\begin{align*}
   i\sigma_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p \\
   i\sigma_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = u(X_2) = (X_2)^p \\
   i\sigma_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = u(X_3) = (X_3)^p
\end{align*}
\]

(5.25)

Because \( \{i\sigma_1, i\sigma_2, i\sigma_3\} \) is a basis of \( \mathfrak{su}(2) \), if we want to know the basis of \( u^{-1}(\mathfrak{so}(2)) = \sqrt[\sqrt{\mathfrak{su}(2)}} \), we have to calculate the \( p \)-th root of \( i\sigma_1, i\sigma_2 \) and \( i\sigma_3 \). Then you can check the validity of the relation:

\[
\begin{align*}
   i\sigma_1 &= \exp \left( \frac{\pi}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\
   i\sigma_2 &= \exp \left( \frac{\pi}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) \\
   i\sigma_3 &= \exp \left( \frac{\pi}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right)
\end{align*}
\]
So with (5.26), it is very easy to calculate the p-th root of \(i\sigma_1, i\sigma_2, i\sigma_3\):

\[
\sqrt[p]{i\sigma_1} = \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)
\]

\[
= \cos\left(\frac{\pi}{2p}\right) \text{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

\[X_1 = \sqrt[p]{i\sigma_1} = \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix}
\]

\[
\sqrt[p]{i\sigma_2} = \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right)
\]

\[
= \cos\left(\frac{\pi}{2p}\right) \text{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]

\[X_2 = \sqrt[p]{i\sigma_2} = \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & i\sin\left(\frac{\pi}{2p}\right) \\ i\sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix}
\]

\[
\sqrt[p]{i\sigma_3} = \exp\left(\frac{\pi}{2p} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)
\]

\[X_3 = \sqrt[p]{i\sigma_3} = \begin{pmatrix} \exp\left(\frac{\pi}{2p}\right) & 0 \\ 0 & \exp\left(-\frac{\pi}{2p}\right) \end{pmatrix}
\]

where \(\text{Id}_2\) is of course the identity 2-dimensional matrix. This Arm-Lie algebra has 6 generators \(i\sigma_1 = (X_1)^p, i\sigma_2 = (X_2)^p, i\sigma_3 = (X_3)^p\) and \(\sqrt[p]{i\sigma_1} = X_1, \sqrt[p]{i\sigma_2} = X_2, \sqrt[p]{i\sigma_3} = X_3\) satisfying the relations:

\[
[X_1, (X_2)^p] = -2 \sin\left(\frac{\pi}{2p}\right) (X_3)^p
\]

\[
[X_2, (X_3)^p] = -2 \sin\left(\frac{\pi}{2p}\right) (X_1)^p
\]

\[
[X_3, (X_1)^p] = -2 \sin\left(\frac{\pi}{2p}\right) (X_2)^p
\]

(5.26)

and

\[
[X_1, X_2] = -2 \sin^2\left(\frac{\pi}{2p}\right) (X_3)^p
\]

\[
[X_2, X_3] = -2 \sin^2\left(\frac{\pi}{2p}\right) (X_1)^p
\]

\[
[X_3, X_1] = -2 \sin^2\left(\frac{\pi}{2p}\right) (X_2)^p
\]

(5.27)

Of course (5.27) and (5.26) imply that the generators of the Arm-Lie algebra \(\sqrt[p]{\text{su}(2)}\) satisfy the u-Jacobi identity. Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie
\[
\exp(\sqrt{\text{su}(2)}) = \left\{ (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 \mid e^{z_1 \psi_1 + z_2 \psi_2 + z_3 \psi_3 + z_4 \psi_4 + z_5 \psi_5 + z_6 \psi_6} \right\} \tag{5.28}
\]

To have an idea of the elements of the group (4.19), we can explicit:

\[
\begin{align*}
\exp(z_1X_1) &= \exp \left( z_1 \cos \left( \frac{\pi}{2p} \right) \right) \begin{pmatrix}
\cos \left( z_1 \sin \left( \frac{\pi}{2p} \right) \right) & -\sin \left( z_1 \sin \left( \frac{\pi}{2p} \right) \right) \\
\sin \left( z_1 \sin \left( \frac{\pi}{2p} \right) \right) & \cos \left( z_1 \sin \left( \frac{\pi}{2p} \right) \right)
\end{pmatrix} \\
\exp(z_3X_3) &= \exp \left( z_3 \cos \left( \frac{\pi}{2p} \right) \right) \begin{pmatrix}
\cos \left( z_3 \sin \left( \frac{\pi}{2p} \right) \right) & i \sin \left( z_3 \sin \left( \frac{\pi}{2p} \right) \right) \\
i \sin \left( z_3 \sin \left( \frac{\pi}{2p} \right) \right) & \cos \left( z_3 \sin \left( \frac{\pi}{2p} \right) \right)
\end{pmatrix} \\
\exp(z_5X_5) &= \exp \left( z_5 \cos \left( \frac{\pi}{2p} \right) \right) \begin{pmatrix}
\exp \left( iz_5 \sin \left( \frac{\pi}{2p} \right) \right) & 0 \\
0 & \exp \left( z_5 \sin \left( -\frac{\pi}{2p} \right) \right)
\end{pmatrix} \tag{5.29}
\end{align*}
\]

and the well-known elements of SU(2):

\[
\begin{align*}
\exp(z_2X_1^p) &= \begin{pmatrix}
\cos(z_2) & -\sin(z_2) \\
\sin(z_2) & \cos(z_2)
\end{pmatrix} \\
\exp(z_4X_2^p) &= \begin{pmatrix}
\cos(z_4) & i \sin(z_4) \\
i \sin(z_4) & \cos(z_4)
\end{pmatrix} \\
\exp(z_6X_3^p) &= \begin{pmatrix}
\exp(iz_6) & 0 \\
0 & \cos(-iz_6)
\end{pmatrix} \tag{5.30}
\end{align*}
\]

Hence we can identify this Arm-Lie algebra to

\[
\exp(\sqrt{\text{su}(2)}) \equiv r \text{SU}(2, \mathbb{C}) = \left\{ M \in M_n(\mathbb{C}) \mid M^\dagger M = r \text{Id}_2 ; \quad r \in \mathbb{C} \right\} \tag{5.31}
\]

for \( \frac{1}{p} \neq 1[2] \) and just SU(2) if \( \frac{1}{p} = 1[2] \).
Discussion

Unfortunately, I searched other Arm-Lie algebras among classics Lie algebras but I didn’t find. I think it’s because many other classic groups are not the exponential of something (i.e. their logarithms do not exist) but I’m not sure. I find this structure only for $\mathfrak{su}(2)$ and its subalgebra $\mathfrak{so}(2)$. I tried with $\mathfrak{so}(3)$ and $\mathfrak{sl}(2)$ but it didn’t work. I develop this theory in order to classify what I found but I hope there is other Lie algebras which are Arm-Lie algebras. There is not a lot of example of Arm-Lie algebra but $\sqrt[\nu]{\mathfrak{su}(2)}$ is a beautiful and very fundamental example.

I also tried this theory for other $u(X)$ for example I took $u(X) = \exp(\exp(X))$ but it is just a shifted case of the traditional case. I was also limited by the choice of changing of variable that I was able to do because I had to find the inverse function of matricial variable which is sometimes hard to do. I also tried function as sin or cos but it gave the identity or zero respectively. The only good changing of variable which I found was $u(X) = X^p$ which is a lot of changing for $p \in \mathbb{C}$ but it finally give the same result for all $p$.

Finally we can imagine an other Lie group theory based on the ‘p-exponential’ which I introduced in [2] but now I think that I will give the same result as the usual Lie group theory but with the p-exponential instead of the traditional exponential. Maybe I will explore this way in an other work.
Références