THE SHORT PROOFS

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Dedicated to my Parents and my Brother

Abstract. The short proofs of the Fermat’s Last Theorem for even \( n \geq 4 \).

I. INTRODUCTION

It is known that for each primitive Pythagorean triple \((x, y, z)\) there exist two relatively prime natural numbers \( u, v \) such that \( u - v \) is positive and odd.

Moreover

\[
x^2 + y^2 = z^2 \land (x + y)^2 + [\pm (x - y)]^2 = 2z^2,
\]

where \( z \) is odd because for all \( a, b \in \mathbb{N} \):

\[
\frac{1}{2} [(2a + 1)^2 + (2b + 1)^2] \text{ is odd.}
\]

II. THE FERMAT’S LAST THEOREM FOR EVEN \( n \)

Theorem 1 (FLT). For all \( n \in \{4, 6, 8, \ldots\} \) the equation

\[
X^n + Y^n = Z^n
\]

has no primitive solutions in \( \mathbb{N}_1 \).

Proof of the Main Theorem. Suppose that for some \( n \in \{4, 6, 8, \ldots\} \) the equation

\[
X^n + Y^n = Z^n
\]

has primitive solutions \([X, Y, Z]\) in \( \mathbb{N}_1 \).

A. The Proof of the Main Theorem for \( 4 \mid n \).

We assume that for some relatively prime natural numbers \( u, v \) such that \( u - v \) is positive and odd:

\[
\begin{bmatrix}
u^2 - v^2 = \left(X^{\frac{n}{4}}\right)^2 \land 2uv = Y^{\frac{n}{2}} \land u^2 + v^2 = Z^{\frac{n}{2}}
\end{bmatrix}.
\]

Thus on the strength of (1):

\[
\begin{bmatrix}
2u^2 = \left(X^{\frac{n}{4}}\right)^2 + \left(Z^{\frac{n}{4}}\right)^2 \land v = X^{\frac{n}{4}} \land X^{\frac{n}{4}} + v = Z^{\frac{n}{2}}
\end{bmatrix} \Rightarrow

\begin{bmatrix}
3X^{\frac{n}{4}} = Z^{\frac{n}{2}} \land X^{\frac{n}{4}} = Z^{\frac{n}{2}}
\end{bmatrix} \Rightarrow \gcd(X, Z) > 1,
\]

which is inconsistent with \( \gcd(X, Z) = 1 \).
B. The Proof of the Main Theorem for $4 \nmid n$

We assume that for some $m \in \{3, 5, 7, \ldots \}$ and for some mutually coprime odd natural numbers $a, b, c, d$ and for some relatively prime natural numbers $u, v$ such that $u - v$ is positive and odd:

\[
\begin{align*}
2m &= n \land (abcd)^{2m} = (u + v)^2 (u - v)^2 = (X^m)^2 = (Z^m + Y^m) (Z^m - Y^m) \land \\
(ab)^m &= u + v \land (cd)^m = u - v \land u^2 + v^2 = Z^m \land 2uv = Y^m \land \\
b^{2m} &= Z + Y \land d^{2m} = Z - Y \land (bd)^{2m} = Z^2 - Y^2 \land 4 \nmid Y,
\end{align*}
\]

which is inconsistent with $4 \nmid Y$ [2]. This is the proof.

[\Box]

III. THE PROOFS OF THE FERMAT’S LAST THEOREM FOR $n = 4$

Theorem 2. The equation

\[Z^4 - Y^4 = x^2\]

has no primitive solutions in $\mathbb{N}_1$.

Proof. Suppose that the equation

\[Z^4 - Y^4 = x^2\]

has the primitive solutions $[Z, Y, x]$ in $\mathbb{N}_1$.

A. The Proof For Odd $x$.

We assume that for some mutually coprime natural numbers $Z, Y, p, q$, where only $Y$ is even:

\[
\begin{align*}
(Z^2 + Y^2 = p^2 \land Z^2 - Y^2 = q^2 [1]) \land 2Z^2 = p^2 + q^2 \land pq = x.
\end{align*}
\]

Thus on the strength of (1):

\[
(p = Y + q \land \pm q = q - Y) \Rightarrow (p = q \lor p = 3q) \Rightarrow \gcd(p, q) > 1,
\]

which is inconsistent with $\gcd(p, q) = 1$. \[\n\]

B. The Proof For Even $x$.

We assume that for some relatively prime natural numbers $u, v$ such that $u - v$ is positive and odd:

\[
(u^2 + v^2 = Z^2 \land u^2 - v^2 = Y^2 \land 2uv = x \land 2u^2 = Z^2 + Y^2).
\]

Thus on the strength of (1):

\[
(Z = Y + v \land \pm Y = v - Y) \Rightarrow (Z = 3Y \lor Z = Y) \Rightarrow \gcd(Z, Y) > 1,
\]

which is inconsistent with $\gcd(Z, Y) = 1$. This is the proof. \[\Box\]

Corollary 1. The equation $Z^4 - Y^4 = x^2$ has no primitive solutions $[Z, Y, X]$ in $\mathbb{N}_1$, where $X = \sqrt{x}$. This is the corollary.

References


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