Why the Composite Magnetic Monopoles of Yang-Mills Gauge Theory have all the Required Chromodynamic and Confinement Symmetries of Baryons, how these may be developed into Topologically-Stable Protons and Neutrons, and how to Analytically Path Integrate the Yang-Mills Action

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Abstract: We develop in detail, the classical magnetic monopoles of non-abelian Yang-Mills gauge theory, and show how these classical monopoles, when analyzed using Gauss' / Stokes' theorem, appear to confine their gauge fields, and also, appear to be composite objects. Of course, baryons, which include the protons and neutrons at the heart of nuclear physics, also confine their gauge fields and are similarly-composite objects. This raises the question whether the magnetic monopoles of Yang-Mills theory are in some fashion related to the observed physical baryons. After developing inverse solutions for the non-abelian electric charge densities while carefully examining uniqueness and gauge fixing, we use these solutions together with Dirac theory to "populate" these classical monopoles with fermions. Applying the Fermi-Dirac-Pauli Exclusion Principle to these fermions forces the selection of a rank-3 gauge group initially chosen to be SU(3). We then find that these non-abelian magnetic monopoles have the exact chromodynamic symmetries of baryons and interact via colored magnetic fields with the exact chromodynamic symmetries of mesons. We show that these monopoles are also topologically stable, and that a required U(1) factor which ensures this stability also "flavors" these monopole as protons and neutrons. Because this exposition is classical, we also discuss the extent to which classical field theory can be used to effectively analyze baryons and confinement. We finally point out how a recursive aspect of the non-abelian electric charge solution may be used to perform an analytically-exact quantum path integration for Yang-Mills theory, proving the existence of a non-trivial quantum Yang–Mills theory on R^4 for any simple gauge group G.

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Contents

1. Introduction: The Field Strength Curvature Tensor in Gauge Theory, and a Review of Gauge-Covariant Derivatives

In 1918, [1], [2] Hermann Weyl first conceived the idea that electrodynamics might be unified with Einstein's recently-developed geometric theory of gravitation [3], by analyzing a "twisting" of vectors under parallel transport to measure the geometric curvature of a gauge space. While Weyl first conceived of this as a local "gauge" symmetry, in 1929 [4] he corrected his original misconception into the modern view of a local "phase" symmetry. Notwithstanding, the original misnomer "gauge" is still used to name Weyl's theory, perhaps as a reminder to posterity that even the most foundational physical theories are sometimes properly-conceived in the abstract but misconceived in some details that need to be worked out over time.

In gravitational theory the Riemann curvature tensor $R^{\sigma}{}_{\alpha\mu\nu}$ may of course be *defined* as a measure of the degree to which the gravitationally-covariant derivative $\partial_{;\mu}$ is non-commuting when it operates on an arbitrary vector A_σ, that is, as $R^{\sigma}{}_{\alpha\mu\nu}A_\sigma \equiv \left[\partial_{;\mu},\partial_{;\nu}\right]A_\alpha$. What Weyl essentially found is that the antisymmetric, second rank, field strength tensor / bivector $F_{\mu\nu}$ which appears in electromagnetic theory may be defined as a measure of the extent to which the gauge-covariant derivative D_μ is not self-commuting when it operates on an arbitrary scalar field φ. That is, $F_{\mu\nu}$ may be *defined* analogously to $R^{\sigma}{}_{\alpha\mu\nu}$, as a type of curvature in "gauge space," by:

$$F_{\mu\nu}\varphi \equiv i\left[D_\mu, D_\nu\right]\varphi = iD_\mu\left(D_\nu\varphi\right) - iD_\nu\left(D_\mu\varphi\right). \tag{1.1}$$

It is instructive to review how the explicit relationship between the field strength $F_{\mu\nu}$ and a gauge / vector potential G_μ then arises from this definition (1.1).

Gauge-covariant derivatives, like covariant derivatives in Riemannian geometry, take a form that depends on the representation of the object they act upon. Taking the gauge field as the defining (fundamental) representation, the form of the gauge-covariant derivatives in (1.1) is $D_\mu = \partial_\mu - iG_\mu$. But in other situations to be reviewed, it is a bit more complicated than this. (In general, for compactness, we scale the interaction charge strength g into the gauge field via $gG_\mu \to G_\mu$. This g can always be extracted back out when explicitly needed.) So, applying $D_\mu = \partial_\mu - iG_\mu$ in (1.1), we may write:

$$\begin{aligned} &iD_\mu\left(D_\nu\varphi\right) = i\left(\partial_\mu - iG_\mu\right)\left(\left(\partial_\nu - iG_\nu\right)\varphi\right) = i\partial_\mu\left(\partial_\nu\varphi - iG_\nu\varphi\right) + G_\mu\left(\partial_\nu\varphi - iG_\nu\varphi\right) \\ &= i\partial_\mu\partial_\nu\varphi + \partial_\mu G_\nu\varphi + G_\nu\partial_\mu\varphi + G_\mu\partial_\nu\varphi - iG_\mu G_\nu\varphi \end{aligned}, \tag{1.2}$$

as well as the reverse-signed, transposed-indexed:

$$-iD_\nu\left(D_\mu\varphi\right) = -i\partial_\nu\partial_\mu\varphi - \partial_\nu G_\mu\varphi - G_\mu\partial_\nu\varphi - G_\nu\partial_\mu\varphi + iG_\nu G_\mu\varphi. \tag{1.3}$$

Using (1.2) and (1.3) in (1.1) then yields:

$$F_{\mu\nu}\varphi \equiv i\left[D_\mu, D_\nu\right]\varphi = iD_\mu\left(D_\nu\varphi\right) - iD_\nu\left(D_\mu\varphi\right) = i\left[\partial_\mu, \partial_\nu\right]\varphi + \partial_{[\mu}G_{\nu]}\varphi - i\left[G_\mu, G_\nu\right]\varphi . \tag{1.4}$$

In flat spacetime where $R^\sigma{}_{\alpha\mu\nu}A_\sigma \equiv \left[\partial_{;\mu}, \partial_{;\nu}\right]A_\alpha = \left[\partial_\mu, \partial_\nu\right]A_\alpha = 0$ and removing the arbitrary operand field φ, the above becomes the more familiar:

$$F_{\mu\nu} = \partial_{[\mu}G_{\nu]} - i\left[G_\mu, G_\nu\right] = \left(\partial_{[\mu} - iG_{[\mu}\right)G_{\nu]} = D_{[\mu}G_{\nu]} . \tag{1.5}$$

Again, $D_\mu \equiv \partial_\mu - iG_\mu$ above is the gauge-covariant derivative when it acts upon gauge field objects G_ν in the fundamental representation, but in general, when operating on other representations, it is a bit more complicated as we shall now see.

If the gauge fields commute, i.e., if $\left[G_\mu, G_\nu\right] = 0$, then (1.5) reduces to $F_{\mu\nu} = \partial_{[\mu}G_{\nu]} = \partial_\mu G_\nu - \partial_\nu G_\mu$ and the gauge theory is known as an *abelian* gauge theory. If the gauge fields do *not* commute, $\left[G_\mu, G_\nu\right] \neq 0$, then (1.5) becomes the field strength for a *non-abelian* gauge theory, often also referred to as Yang-Mills [5] gauge theory.

Using differential forms, we may write the abelian field strength as:

$$F = \tfrac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu = \tfrac{1}{2!}\partial_{[\mu}G_{\nu]}dx^\mu \wedge dx^\nu = \partial_\mu G_\nu dx^\mu \wedge dx^\nu = dG . \tag{1.6}$$

In general, the wedge product $dx^\mu \wedge dx^\nu = dx^\mu dx^\nu - dx^\nu dx^\mu = \left[dx^\mu, dx^\nu\right]$ is antisymmetric under adjacent index interchange, and the differential elements are anticommuting, $dx^\mu dx^\nu = -dx^\nu dx^\mu$. So, by inspection from (1.5) in view of (1.6), the non-abelian field strength is:

$$F = \tfrac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu = \tfrac{1}{2!}\left(\partial_{[\mu}G_{\nu]} - i\left[G_\mu, G_\nu\right]\right)dx^\mu \wedge dx^\nu = dG - i\left[G, G\right] \equiv DG . \tag{1.7}$$

Here, compacted into differential forms, the gauge-covariant derivative is not separable from its operand as was $D_\mu = \partial_\mu - iG_\mu$ when operating on G_ν in (1.1) to (1.5), but rather involves the commutator of G with the operand which, in this case, just so happens to also be G. That is, it involves $\left[G, G\right]$. This in fact reveals the more-general form of the gauge-covariant derivative as we shall review next.

Now, focusing on non-abelian gauge theories, we introduce a set of traceless Hermitian generators $t^i = t^{\dagger i}$ which form a closed group under multiplication via $\left[t^i, t^j\right] = if^{ijk}t^k$, where f^{ijk} are the group structure constants and are antisymmetric under the transposition of any two adjacent indexes. For any simple group SU(N), the internal symmetry indexes of the adjoint representation $i, j, k = 1 ... N^2 - 1$. We may then define $F_{\mu\nu} \equiv t^k F^k{}_{\mu\nu}$ and $G_\mu \equiv t^i G^i{}_\mu$ and use these in (1.5) to expand:

$$F_{\mu\nu} = t^k F^k{}_{\mu\nu} = \partial_{[\mu} G_{\nu]} - i\left[G_\mu, G_\nu\right] = t^k \partial_{[\mu} G^k{}_{\nu]} - i\left[t^i, t^j\right] G^i{}_\mu G^j{}_\nu = t^k \partial_{[\mu} G^k{}_{\nu]} + f^{ijk} t^k G^i{}_\mu G^j{}_\nu . \quad (1.8)$$

Factoring out t^k this simplifies to the recognizable:

$$F^k{}_{\mu\nu} = \partial_{[\mu} G^k{}_{\nu]} + f^{ijk} G^i{}_\mu G^j{}_\nu . \quad (1.9)$$

Now, for illustration, let us momentarily consider the situation where the t^i are one half (½) times the three (3) Pauli spin matrix generators of SU(2), $t^i = \frac{1}{2}\sigma^i$, so that f^{ijk} simply becomes the rank-3 Levi-Civita tensor, $f^{ijk} \to \varepsilon^{ijk}$, which again, is antisymmetric in all indexes. In spacetime, if we were to write $\varepsilon^{ijk} A^i B^j$ for any two vectors A^i and B^j and were to regard i, j, k as indexes for the *space* dimensions *x*, *y*, *z*, then, for example, $\varepsilon^{ij3} A^i B^j = A^1 B^2 - A^2 B^1 = (\mathbf{A} \times \mathbf{B})^3$ is the z-component of the cross product $\mathbf{A} \times \mathbf{B}$, and more generally, $\varepsilon^{ijk} A^i B^j = (\mathbf{A} \times \mathbf{B})^k$. But of course, the i, j, k indexes in (1.9) are not space indexes, but are *internal symmetry* indexes. So rather than using the cross-product symbol "$\times$" which is used for vectors in physical space, and because we still wish to be able compactly represent the fundamentally-antisymmetric character of f^{ijk} in the form of a "cross-like product" in internal symmetry space, we instead employ the wedge symbol "$\wedge$." Although $G^i{}_\mu$ and $G^j{}_\nu$ in (1.9) both are gauge fields *G*, they have different spacetime indexes μ and ν, so we may still think of them as two different vectors just like A^i and B^j above. So analogously to $\varepsilon^{ijk} A^i B^j = (\mathbf{A} \times \mathbf{B})^k$ in the three space dimensions of spacetime, we write $f^{ijk} G^i{}_\mu G^j{}_\nu = \left(G_\mu \wedge G_\nu\right)^k$ in internal symmetry space. Then, we use this in (1.9) to write $F^k{}_{\mu\nu} = \partial_{[\mu} G^k{}_{\nu]} + \left(G_\mu \wedge G_\nu\right)^k$. Because the general form of this equation holds in SU(N) for each of the indexes $k = 1 ... N^2 - 1$, we may suppress the *k* index throughout to write:

$$F_{\mu\nu} = \partial_{[\mu} G_{\nu]} + G_\mu \wedge G_\nu . \quad (1.10)$$

Then, compacting (1.10) to differential forms as in (1.6), we have:

$$F = \tfrac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = \tfrac{1}{2!}\left(\partial_{[\mu} G_{\nu]} + G_\mu \wedge G_\nu\right) dx^\mu \wedge dx^\nu = dG + G \wedge G = (d + G \wedge) G \equiv DG . \quad (1.11)$$

Now, Jaffe and Witten point out at pages 1 and 2 of [6], that:

> "If *A* denotes the U(1) gauge connection, locally a one-form on space-time, then the curvature or electromagnetic field tensor is the two-form $F = dA$ [see (1.6)

> above], and Maxwell's equations in the absence of charges and currents read $0 = dF = d*F$."

They then proceed to explain that in "non-abelian gauge theory":

> "at the classical level one replaces the gauge group U(1) of electromagnetism by a compact gauge group G. The definition of the curvature arising from the connection must be modified to $F = dA + A \wedge A$ and Maxwell's equations are replaced by the Yang–Mills equations, $0 = d_A F = d_A * F$, where d_A is the gauge-covariant extension of the exterior derivative."

Equation (1.11) is precisely $F = dA + A \wedge A$ with the gauge field simply renamed from *A* to *G*, and what Jaffe and Witten write above is a condensed explanation for what we have laid out above in equations (1.1) through (1.11). When we use the generalized one-form *G* and two-form *F* without any particular generator set t^i, then the differential forms equation is written as $F = dG - i[G,G]$ in (1.7). But when one does introduce a set of group generators t^i and the antisymmetric structure contestants $f^{ijk} \to \wedge$, the differential forms equation is $F = dG + G \wedge G$ in (1.11). To display the particular $i = 1 ... N^2 - 1$ field components for a compact simple gauge group SU(N), this equation is $F^i = dG^i + (G \wedge G)^i$. So $F = dG - i[G,G]$ (commutator form) and $F = dG + G \wedge G$ (wedge form) are just alternative ways of saying the same thing. But a benefit of the wedge form is that we may write $F = (d + G \wedge)G \equiv DG$ so as to define a gauge-covariant derivative $D \equiv (d + G \wedge) \quad (= d_A)$ in a form which is fully-separable from its operand, and which is generally applicable to *any and all operands*. We will find it useful in general to develop both these forms.

Indeed, the reason we have gone through the exercise of (1.8) through (1.11), is to explore the question of how one generally performs $d_A = D$, independently of its operand, "where d_A is the gauge-covariant extension of the exterior derivative." That is, we want to be able to generalize the taking of these derivatives, and especially, to ascertain the correct way to derive the equations $*J = d_A * F = D * F$ and $P = d_A F = DF$ *in the presence of* the electric and magnetic three-form charge densities $*J$ and P.

Specifically, as already stated, if we write equation (1.11) as $F = (d + G \wedge)G \equiv DG$ with $D \equiv (d + G \wedge)$, we find that $D \equiv (d + G \wedge)$ *is in fact the generalized definition of the gauge-covariant derivative which tells us how to take higher-rank gauge derivatives, independent of the representation of the operand*. Thus, the Maxwell equations for Yang-Mills theory, in differential forms, where t^i and f^{ijk} are specified, with index *i* suppressed, for SU(N), where we use the duality operator *, and with $F = dG + G \wedge G$, are merely the $i = 1 ... N^2 - 1$ equations:

$$\begin{aligned}
{}^*J &= D{}^*F = D{}^*DG = \left(d + G\wedge\right){}^*F = d{}^*F + G\wedge{}^*F = d{}^*\left(dG + G\wedge G\right) + G\wedge{}^*\left(dG + G\wedge G\right) \\
&= d{}^*dG + d{}^*\left(G\wedge G\right) + G\wedge{}^*dG + G\wedge{}^*\left(G\wedge G\right) \\
P &= DF = DDG = \left(d + G\wedge\right)F = dF + G\wedge F = d\left(dG + G\wedge G\right) + G\wedge\left(dG + G\wedge G\right) \\
&= ddG + d\left(G\wedge G\right) + G\wedge dG + G\wedge G\wedge G
\end{aligned} \quad .(1.12)$$

The duality operator * was first developed by Reinich [7] later elaborated by Wheeler [8], and it makes integral use of the Levi-Civita tensor as laid out in [9] at pages 87-89.

In this paper, we shall develop the classical Yang-Mills magnetic monopole density P and a related "faux" magnetic charge density P' in detail, and shall show how the related monopole density P', when analyzed using Gauss' / Stokes' theorem, appears to confine its gauge fields. Of course, baryons, which include the protons and neutrons at the heart of nuclear physics, also confine their gauge fields. So this raises the question which we thereafter explore in detail, whether the magnetic monopoles of Yang-Mills theory are in some fashion related to baryons.

2. Classical Field Equations for the Yang-Mills Magnetic Monopole

To further develop the monopole density P, first, akin to the derivation (1.1) through (1.5), we calculate the commutator:

$$\begin{aligned}
&\left[D_\sigma, F_{\mu\nu}\right]\varphi = D_\sigma\left(F_{\mu\nu}\varphi\right) - F_{\mu\nu}D_\sigma\varphi = \left(\partial_\sigma - iG_\sigma\right)\left(F_{\mu\nu}\varphi\right) - F_{\mu\nu}\left(\partial_\sigma - iG_\sigma\right)\varphi \\
&= \partial_\sigma F_{\mu\nu}\varphi + F_{\mu\nu}\partial_\sigma\varphi - iG_\sigma F_{\mu\nu}\varphi - F_{\mu\nu}\partial_\sigma\varphi + iF_{\mu\nu}G_\sigma\varphi = \partial_\sigma F_{\mu\nu}\varphi - i\left[G_\sigma, F_{\mu\nu}\right]\varphi
\end{aligned} \quad . \quad (2.1)$$

We *can* use $D_\sigma = D_{F\sigma} = \partial_\sigma - iG_\sigma$ in the above, precisely because this is a commutator, and so the gauge field will be commuted with the operand $F_{\mu\nu}$ as in $F = dG - i[G, G]$ a.k.a. $F = dG + G\wedge G$. Removing φ we see that (2.1) contains the useful identity:

$$\left[D_\sigma, F_{\mu\nu}\right] = \partial_\sigma F_{\mu\nu} - i\left[G_\sigma, F_{\mu\nu}\right] = D_\sigma F_{\mu\nu}, \quad (2.2)$$

with the commutator included in the gauge-covariant derivative. Then, combining (2.2) with (1.1) in the form $F_{\mu\nu} = i\left[D_\mu, D_\nu\right]$ first yields:

$$D_\sigma F_{\mu\nu} = \left[D_\sigma, F_{\mu\nu}\right] = i\left[D_\sigma, \left[D_\mu, D_\nu\right]\right] \quad (2.3)$$

containing an anticommuting succession of gauge-covariant derivatives. This in turn means that the index-cyclical combination:

$$P_{\sigma\mu\nu} = D_\sigma F_{\mu\nu} + D_\mu F_{\nu\sigma} + D_\nu F_{\sigma\mu} = i\left(\left[D_\sigma, \left[D_\mu, D_\nu\right]\right] + \left[D_\mu, \left[D_\nu, D_\sigma\right]\right] + \left[D_\nu, \left[D_\sigma, D_\mu\right]\right]\right) = 0, \quad (2.4)$$

by the Jacobian identity $\left[a,\left[b,c\right]\right]+\left[b,\left[c,a\right]\right]+\left[c,\left[a,b\right]\right]=0$. So we see that the *Yang-Mills magnetic monopole densities vanish, just like those of abelian gauge theory*. Consequently, we can append $P=0$ from (2.4) to (1.12), and so write $P=DF=DDG=0$, which is the non-abelian analog to the abelian $ddG=0$.

But there is another zero in the monopole P of (1.12), and that is the zero which comes from this very same abelian $ddG=0$. This is rooted in the geometric relationship $dd=\mathbf{0}$ of exterior calculus in spacetime: "the exterior derivative of an exterior derivative is zero." In general in this paper, we shall highlight the zero of $dd=\mathbf{0}$ to distinguish it from the (not highlighted) zero of the Jacobian identity $DDG=0$ which is established by the combination of (1.12) and (2.4). The highlighted zero in $dd=\mathbf{0}$ is a "subset" identity contained within (1.12), which we may now rewrite as:

$$\begin{aligned}0=P=DF=DDG&=ddG+d\left(G\wedge G\right)+G\wedge dG+G\wedge G\wedge G\\&=\mathbf{0}+d\left(G\wedge G\right)+G\wedge dG+G\wedge G\wedge G\end{aligned}. \tag{2.5}$$

Of course, in an abelian gauge theory such as Maxwell's electrodynamics where $\left[G_\mu,G_\nu\right]=0$ so that $F_{\mu\nu}=\partial_{[\mu}G_{\nu]}$ in (1.5) thus $F=dG$, the Magnetic monopole densities are themselves specified by $P_{\text{abelian}}=dF=ddG=\mathbf{0}$. This means that the Yang-Mills monopole density in (2.5), although it too is equal to zero, contains a number of non-zero terms embedded within, as well as the term $ddG=\mathbf{0}$ which we associate with the vanishing monopoles of electrodynamics. This will be very important to keep in mind as we develop this monopole, because this "abelian subset" embedding of $ddG=\mathbf{0}$ within (2.5) will be directly responsible for *confining* the gauge fields within the Yang-Mills monopole, and will lead us to consider whether there is some connection between Yang-Mills monopoles and baryons.

Next let us ascertain the commutator form for the monopole (2.5). Via the exact same type of calculation we used to turn (1.5) a.k.a. (1.7) into (1.11), one may demonstrate that $P=DF=dF-i\left[G,F\right]$ is equivalent to $P=DF=\left(d+G\wedge\right)F$. So, combining the former, $P=DF=dF-i\left[G,F\right]$, with $F=DG=dG-i\left[G,G\right]$ from (1.7) a.k.a. $F=DG=\left(d+G\wedge\right)G$ from (1.11) , we may translate (2.5) into the commutator expression:

$$\begin{aligned}P=DF=DDG&=dF-i\left[G,F\right]=d\left(dG-i\left[G,G\right]\right)-i\left[G,dG-i\left[G,G\right]\right]\\&=ddG-id\left[G,G\right]-i\left[G,dG\right]-\left[G,\left[G,G\right]\right]\\&=\quad\mathbf{0}-id\left[G,G\right]-i\left[G,dG\right]-\left[G,\left[G,G\right]\right]=0\end{aligned}. \tag{2.6}$$

Let us now expand (2.6) above into tensor components term-by-term, and then do some additional reductions. For P and $-id\left[G,G\right]$ we have:

$$P = \tfrac{1}{3!} P_{\sigma\mu\nu} dx^\sigma \wedge dx^\mu \wedge dx^\nu = P_{\sigma\mu\nu} dx^\sigma dx^\mu dx^\nu \,, \tag{2.7}$$

$$\begin{aligned}
& -id\left[G,G\right] = -\tfrac{1}{3!} i\left(\partial_\sigma\left[G_\mu,G_\nu\right] + \partial_\mu\left[G_\nu,G_\sigma\right] + \partial_\nu\left[G_\sigma,G_\mu\right]\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -\tfrac{1}{2!} i\partial_\sigma\left[G_\mu,G_\nu\right] dx^\sigma \wedge dx^\mu \wedge dx^\nu = -i\partial_\sigma\left(G_\mu G_\nu\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\left(\partial_\sigma G_\mu G_\nu + G_\mu \partial_\sigma G_\nu\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu = \left(-i\partial_\sigma G_\mu G_\nu + iG_\sigma \partial_\mu G_\nu\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -idGG + iGdG
\end{aligned} \quad . \tag{2.8}$$

The sign reversal in the third line of (2.8) reveals the identity $d\left[G,G\right] = dGG - GdG$, in contrast to scalar product rule $d\left(a \cdot b\right) = da \cdot b + a \cdot db$. For $-i\left[G,dG\right]$ in (2.6) we further have:

$$\begin{aligned}
& -i\left[G,dG\right] = -\tfrac{1}{3!} i\left(\left[G_\sigma,\partial_{[\mu}G_{\nu]}\right] + \left[G_\mu,\partial_{[\nu}G_{\sigma]}\right] + \left[G_\nu,\partial_{[\sigma}G_{\mu]}\right]\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -\tfrac{1}{2!} i\left[G_\sigma,\partial_{[\mu}G_{\nu]}\right] dx^\sigma \wedge dx^\mu \wedge dx^\nu = -i\left[G_\sigma,\partial_\mu G_\nu\right] dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\left(G_\sigma \partial_\mu G_\nu - \partial_\mu\left(G_\nu G_\sigma\right)\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\left(G_\sigma \partial_\mu G_\nu - G_\nu \partial_\mu G_\sigma - \partial_\mu G_\nu G_\sigma\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= \left(-2iG_\sigma \partial_\mu G_\nu + i\partial_\sigma G_\mu G_\nu\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -2iGdG + idGG
\end{aligned} \tag{2.9}$$

in which the GdG doubles by a similar sign reversal in the fifth line. Finally, by the Jacobian identity $\left[a,\left[b,c\right]\right] + \left[b,\left[c,a\right]\right] + \left[c,\left[a,b\right]\right] = 0$, for $\left[G,\left[G,G\right]\right]$ in (2.6), we find (cf. (2.4)) that:

$$-\left[G,\left[G,G\right]\right] = -\tfrac{1}{3!}\left(\left[G_\sigma,\left[G_\mu,G_\nu\right]\right] + \left[G_\mu,\left[G_\nu,G_\sigma\right]\right] + \left[G_\nu,\left[G_\sigma,G_\mu\right]\right]\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu = 0 \,, \tag{2.10}$$

In (2.6), we then use $-id\left[G,G\right] = -idGG + iGdG$ and $-i\left[G,dG\right] = -2iGdG + idGG$ and $-\left[G,\left[G,G\right]\right] = 0$ from (2.8) to (2.10) in (2.6) to restructure and consolidate the monopole density as much as possible while retaining an Gauss / Stokes integrable $d\left[G,G\right]$ term, into:

$$\begin{aligned}
P &= \mathbf{0} - id\left[G,G\right] - i\left[G,dG\right] \\
&= \mathbf{0} - idGG + iGdG - 2iGdG + idGG \\
&= \mathbf{0} - iGdG \\
&= -\mathbf{0} + id\left[G,G\right] - idGG = 0
\end{aligned} \quad . \tag{2.11}$$

This in turn reveals the additional identities $d\left[G,G\right] = dGG$ and $GdG = 0$.

Now, of central interest in the discussion to follow, the monopole density in the final line above contains a Gauss/Stokes-integrable term $d\left[G,G\right]$ (and the $\mathbf{0}=ddG$) together with the non-integrable term dGG. Applying Gauss' / Stokes Theorem $\iint dX = \oint X$ for any differential form X to the final line above, we may ascertain the classical surface flux associated with this non-abelian magnetic monopole, namely:

$$\begin{aligned}\iiint P &= \iiint\left(-ddG+id\left[G,G\right]-idGG\right)=\iiint\left(-\mathbf{0}+id\left[G,G\right]-idGG\right)\\ &=-\oiint dG+i\oiint\left[G,G\right]-i\iiint dGG=-\mathbf{0}+i\oiint\left[G,G\right]-i\iiint dGG=0\end{aligned}. \tag{2.12}$$

By then writing (2.12) using the not-highlighted 0 of the Jacobian identity (2.4) as:

$$\begin{aligned}-\oiint dG+i\oiint\left[G,G\right]&=i\iiint dGG\\ -\mathbf{0}+i\oiint\left[G,G\right]&=i\iiint dGG\end{aligned}, \tag{2.13}$$

we clearly see the relationship between what is contained within the three-dimensional volume $\iiint$ and what net flows through the closed two-dimensional surface $\oiint$ enclosing that volume. Now, we wish to interpret what is being taught by (2.13).

3. Confinement of Gauge fields within, and the Composite Nature of, Yang-Mills Magnetic Monopoles

We start with the term $\oiint dG=\mathbf{0}$ which is embedded in (2.13). In electrodynamics, Gauss' law for magnetism and Faraday's law are both contained within:

$$\iiint P=\iiint dF=\iiint ddG=\oiint F=\oiint F^{\mu\nu}dx_\mu dx_\nu=\oiint dG=\mathbf{0}. \tag{3.1}$$

At rest, this tells us that while magnetic fields may flow across some surfaces, there is never a *net* flux of a magnetic field through any *closed* two dimensional surface. In the form $P=dF=ddG=\mathbf{0}$, this simply says there are no observed magnetic charges. So how might we interpret the presence of $\oiint dG=\mathbf{0}$ as *one of the terms* among a number of *non-vanishing* terms in equations (2.12) and (2.13) for the Yang-Mills magnetic monopoles?

To find out, let us return to the *non-abelian*, *Yang-Mills* field strength (1.5), namely $F_{\mu\nu}=\partial_{[\mu}G_{\nu]}-i\left[G_\mu,G_\nu\right]$, and rewrite this using the differential forms equation:

$$\begin{aligned}\oiint F&=\tfrac{1}{2!}\oiint F_{\mu\nu}dx^\mu\wedge dx^\nu=\oiint\partial_\mu G_\nu dx^\mu\wedge dx^\nu-\tfrac{1}{2!}i\oiint\left[G_\mu,G_\nu\right]dx^\mu\wedge dx^\nu\\ &=\oiint dG-i\oiint\left[G,G\right]=\mathbf{0}-i\oiint\left[G,G\right]\end{aligned}. \tag{3.2}$$

We may then use (3.2) to rewrite (2.13) with a sign reversal as:

$$\begin{aligned}\oiint F &= -i\oiint [G,G] = -i\iiint dGG \\ &= -i\iiint \tfrac{1}{3!}\left(\partial_\sigma\left[G_\mu,G_\nu\right]+\partial_\mu\left[G_\nu,G_\sigma\right]+\partial_\nu\left[G_\sigma,G_\mu\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu \; . \\ &= -i\iiint \tfrac{1}{3!}\left(\partial_{[\sigma}G_{\mu]}G_\nu+\partial_{[\mu}G_{\nu]}G_\sigma+\partial_{[\nu}G_{\sigma]}G_\mu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu \neq 0\end{aligned} \tag{3.3}$$

So, while (3.1) tells us that there is no net magnetic field flux over of any closed surface in *abelian* electrodynamics, (3.3) tells us that in non-Abelian, Yang-Mills gauge theory, there is indeed a *non-vanishing* net flux across closed surfaces, $\oiint F \neq 0$, of *whatever the Yang-Mills analog is to an ordinary abelian magnetic field*.

Now, we have a puzzle: any time we see a term $\oiint F$, we know that we are talking about a magnetic monopole, and that whatever is contained within the associated volume integral is a magnetic charge. Indeed, (3.3) may be thought of as *the very definition of a magnetic charge*, which in (3.3) is *not* zero. At the same time, we found in (2.4) a.k.a. (2.6) that $P = DF = DDG = 0$, which is to say, that the magnetic charge density is zero, just as it is in electrodynamics. So if $P = DF = DDG = 0$ but $\oiint F \neq 0$, how do we reconcile the former equation which says the magnetic charge density is zero with the latter equation which says there is a non-zero magnetic charge?

One way to think this through, is take the Yang-Mills electric charge field equation (1.12), $*J = D*F$, revert this (merely for pedagogic simplicity) to its abelian form $*J = d*F$ which contains Gauss' law for electricity, and then apply Gauss' / Stokes' Theorem to obtain $\oiint *F = \iiint *J \quad \left(= \iiint d*F\right)$. Just as $\oiint F$ in the rest frame represents a net flux of magnetic field through a closed surface, $\oiint *F$ in the rest frame represents a net flux of electric field through a closed surface. And this $\oiint *F$ then becomes the very definition of the *electric* charge. But here, electric charge density is defined by $*J$ inside $\iiint *J$, while in (3.3) magnetic charge density is defined by $-idGG$ inside $-i\iiint dGG$. That is, we have a magnetic charge density $-idGG$ which we need to think about in comparison to an electric charge density $*J$.

The answer to this puzzle is that the magnetic charge density in (3.3) is *not* the P of $P = DF = DDG = 0$, it is the $P' \equiv -idGG$ which, via (2.11) can be extended to $P' = -id[G,G] = -idGG$. The magnetic charge as defined by the enclosure surface $\oiint F$ is a three-form just like $*J$ and P, but it is not an *elementary* three-form source. Rather, it is a three-form constructed from $-idGG$ which includes some dynamical behavior of the gauge fields inside the volume integral. That is, the magnetic charge $P' = -id[G,G] = -idGG$ is a *composite three-form* built out of gauge fields, rather than an elementary three form like the abelian electric charge source $*J$. Indeed, we may take this a step further:

In electrodynamics, the three-form $*J$ which in tensor language is related to the electric source current density vector J^α by $*J_{\sigma\mu\nu} = (-g)^{.5}\,\varepsilon_{\alpha\sigma\mu\nu} J^\alpha$, is a *true electric source* which then gives rise to gauge fields in abelian gauge theory via $*J = d*F = d*dG$, and per (1.12), via $*J = D*F = D*DG$ in Yang-Mills gauge theory. On the other hand, the $P' = -idGG$ in (3.3), written in tensor form as $P'_{\sigma\mu\nu} = -i\left(\partial_{[\sigma}G_{\mu]}G_\nu + \partial_{[\mu}G_{\nu]}G_\sigma + \partial_{[\nu}G_{\sigma]}G_\mu\right)$ and converted over to a one form via the related general identities $*P'^\alpha = \frac{1}{3!}(-g)^{-.5}\,\varepsilon^{\sigma\mu\nu\alpha}P'_{\sigma\mu\nu}$ and $*\partial^{[\nu}G^{\alpha]} = \frac{1}{2}(-g)^{-.5}\,\varepsilon^{\sigma\mu\nu\alpha}\partial_{[\sigma}G_{\mu]}$, will result in a *faux magnetic source*:

$$
\begin{aligned}
*P'^\alpha &= \tfrac{1}{3!}(-g)^{-.5}\,\varepsilon^{\sigma\mu\nu\alpha}P'_{\sigma\mu\nu} = -(-g)^{-.5}\,\varepsilon^{\sigma\mu\nu\alpha}\tfrac{1}{3!}i\left(\partial_{[\sigma}G_{\mu]}G_\nu + \partial_{[\mu}G_{\nu]}G_\sigma + \partial_{[\nu}G_{\sigma]}G_\mu\right)\\
&= -(-g)^{-.5}\tfrac{1}{3}i\left(\tfrac{1}{2}\varepsilon^{\sigma\mu\nu\alpha}\partial_{[\sigma}G_{\mu]}G_\nu + \tfrac{1}{2}\varepsilon^{\sigma\mu\nu\alpha}\partial_{[\mu}G_{\nu]}G_\sigma + \tfrac{1}{2}\varepsilon^{\sigma\mu\nu\alpha}\partial_{[\nu}G_{\sigma]}G_\mu\right)\\
&= -\tfrac{1}{3}i\left(*\partial^{[\nu}G^{\alpha]}G_\nu + *\partial^{[\sigma}G^{\alpha]}G_\sigma + *\partial^{[\mu}G^{\alpha]}G_\mu\right)\\
&= -i*\partial^{[\sigma}G^{\alpha]}G_\sigma
\end{aligned}
\quad . \qquad (3.4)
$$

which is constructed solely out of gauge fields G *which themselves are sourced by* $*J = D*F = D*DG$. So, there is only one *elementary* source J, not two sources J and P. From this one source J, gauge fields G are emitted from interaction vertices. From these gauge fields G, a faux magnetic source $P' = -idGG$ is assembled. And finally, from this faux magnetic source, $\oiint F \neq 0$ flows across closed surfaces as in (3.3). The electric source J^α, whether in abelian or non-abelian gauge theory, has its own independent existence, and it is the source of any and all gauge fields. But the faux magnetic source charge in (3.3) has *no independent existence* apart from the gauge fields G. Rather, it is built out of the gauge fields. *So the Yang-Mills monopoles are composite, not elementary, objects.* And, by the way, so too are baryons.

Having resolved the puzzle of how to reconcile $P = DF = DDG = 0$ with $\oiint F \neq 0$, we next pose the following question: what happens to the total flux $\oiint F$ in (3.2) under the local gauge-like transformation $F^{\mu\nu} \to F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu}G^{\mu]}$? In differential forms, this transformation is $F \to F' = F - dG$, which means, precisely because $\oiint dG = \mathbf{0}$, that:

$$\oiint F \to \oiint F' = \oiint(F - dG) = \oiint F\,, \qquad (3.5)$$

So, the net surface flux in the monopole equation (3.3) is *invariant* under the transformation $F^{\mu\nu} \to F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu}G^{\mu]}$, which means that the gauge field is *not observable* with respect to net flux across closed surfaces of the monopole. The abelian expression $\oiint dG = \mathbf{0}$, expanded to show the Riemann tensor, may be written as $\oiint F = \oiint dG = \iiint R^\tau{}_{\nu\sigma\mu}G_\tau dx^\sigma dx^\mu dx^\nu = \mathbf{0}$, and explicitly shows how individual gauge fields G_τ couple with spacetime geometry as represented by $R^{\tau}{}_{\nu\sigma\mu}$. This represents an *absence* of monopoles in electrodynamics, and yields the *symmetry principle* (3.5) for the behavior of magnetic monopoles in Yang-Mills theory generally.

But if the non-zero flux in the Yang-Mills monopole equation (3.3) is invariant under the gauge-like transformation $F^{\mu\nu} \rightarrow F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu}G^{\mu]}$ which means that the gauge fields G^{μ} are not net observables over a closed monopole surface, *this would seem to suggest that the Yang-Mills monopole inherently confine their gauge fields*. This is another hint that the monopole equation (3.3) could be the classical field equation for a baryon, in integral form.

The final point is that because the *faux* magnetic source $P' = -idGG$ is constructed out of gauge fields, and because the gauge fields are in turn sourced by $*J = D*F = D*DG$, and because electric sources may be represented in vector form in terms of Dirac fermion wavefunctions ψ via $J^{\mu} = \overline{\psi}\gamma^{\mu}\psi$, it should be possible in principle, and would certainly be desirable in practice, to rewrite the *faux* magnetic source $-idGG$ in terms of the *true* source currents J^{μ} from which they arise, and then to rewrite the $J^{\mu} = \overline{\psi}\gamma^{\mu}\psi$ in terms of their fermion wavefunctions ψ. The upshot of all this, is that while $\oiint F$ in (3.3) is presently expressed in terms of gauge fields as $\oiint F(G)$, once we obtain the gauge fields $G(J)$ in terms of sources and the sources $J(\psi)$ in terms of fermions, we will end up with $\oiint F\left(G\left(J(\psi)\right)\right)$. Then, if we happen to find more than one fermion (maybe even three fermions) within the enclosed $\oiint F$ "system" in its "ground" state, we would need to apply the Exclusion Principle of Fermi-Dirac-Pauli statistics to maintain the ψ in distinct quantum eigenstates, which would give us the opportunity, for example, to introduce a color degree of freedom to do so and thus make a connection to SU(3)$_C$ Chromodynamics, with $\oiint F\left(G\left(J(\psi_R,\psi_G,\psi_B)\right)\right)$. So this means that the Yang-Mills monopoles are not only composite objects, but are composite objects which contain fermions and gauge fields, and that these fermions will need to obey some form of quantum exclusion which may include SU(3)$_C$. And, by the way, all of the same the same is true of baryons, and as to fermion exclusion, quarks.

It is for these reasons, that it may be fruitful to entertain the prospect that (3.3) is not only the classical field equation for a Yang-Mills magnetic monopole, but may be synonymous with the classical field equation for a baryon. All of the development in sections 5 through 10 serves the singular purpose of proving that this is true. But first, we need to discuss whether a classical analysis along the lines of (3.3) can really teach us anything useful about baryons and confinement.

4. Can a Classical Field Equation Really Teach us Anything Useful about Baryons and Confinement?

Given that (3.3) is a classical field equation, we must pose the question whether such a classical equation can really have anything of interest to say about baryons and confinement, which have many features that arise only out of quantum field theory. For example, it might be observed that a classical analysis which seeks to understand baryons and confinement in no way takes account of quantum field theory with operator-valued fields. This, it might be argued, is despite the fact that there are many reasons to believe confinement and the existence of a mass gap are related to the running of the coupling constant, which is an inherently quantum effect.

Certainly, (3.3) above is a completely classical field equation, not yet taking into account any aspects (or the need to prove existence) of a non-trivial relativistic quantum Yang–Mills theory on $\Re^4$ [6]. And, of course, there are many reasons to believe that confinement is related to the running of the strong coupling constant, which is an inherently quantum effect, and which manifests in asymptotic freedom at "ultraviolet" energy and infrared slavery at low energy [10]. However, just like electrodynamics, Yang-Mills gauge theory has a classical formulation and (is expected once quantum Yang-Mills existence is proven, to have) a quantum field formulation. This means that (3.3) may reveal inherently-confining attributes for the magnetic monopoles of Yang-Mills gauge theory which appear at the classical level and which are rooted in the relationship $dd=\mathbf{0}$ of Riemannian spacetime exterior geometry, as well as inherently-composite attributes expressed by $\oiiint F\left(G\left(J\left(\psi\right)\right)\right)$. That opens up the question how these same attributes translate through to quantum Yang-Mills theory.

Specifically, *if* in fact (3.3) for $\oiiint F$ is an equation for baryon-like gauge field confinement properties of Yang-Mills magnetic monopoles based upon their abelian-subset behaviors rooted in the classical equation $ddG=\mathbf{0}$ and its integral form $\oiiint dG=\mathbf{0}$ and the consequent symmetry (3.5), and if the composite faux magnetic charge $P'=-idGG$ in (3.3) in some way represents a baryon charge, then the classical baryons that would be represented by (3.3) would not suddenly become "not baryons" in quantum field theory. Rather, there would *two sets of behaviors* that need to be studied: a) how these monopoles behave in a classical formulation, which includes (3.3) and (3.5) above, and b) how these monopoles additionally behave in quantum field theory. So if we can demonstrate that the classical behaviors appear to be confining and appear to involve a non-elementary, composite charge that includes some amalgam of fermions and gauge fields, one should expect that this will "bleed" through to yield quantum amplitudes and running couplings and color symmetries that buttress, not defy, these classical behaviors, just as abelian magnetic monopoles do not suddenly appear and ordinary magnetic fields do not suddenly net flow through closed surfaces, once one goes from classical to quantum electrodynamics.

Further, one might take the perspective that the *cause* for confinement and baryon compositeness is the classical field equation (3.3) for a Yang-Mills monopole which has the symmetry (3.5), and that one of the *effects* of this is that in a quantum field treatment of these baryon monopoles, the strong coupling will weaken for ultraviolet and strengthen for infrared probes. And, it can be argued that this is a more natural approach than simply trying to figure out how to "glue" together disparate quarks into baryons *without knowing to begin with what sorts of covariant objects baryons actually are in spacetime*. Indeed, if the hints of baryons and confinement that arise in (3.3) and (3.5) are correct, then we would need to start thinking of baryons as third-rank antisymmetric tensors and related three-forms in spacetime governed by the classical equation (3.3) with the symmetry (3.5), and then see how that connects to everything else we know about baryons. The "let's glue together the quarks" approach, notwithstanding many opportunities to do so, has thus far failed to explain why QCD "must have 'quark confinement, that is, even though the theory is described in terms of elementary fields, such as the quark fields, that transform non-trivially under SU(3), the physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant," see [6] at page 3. This SU(3)-invariance of *physical particle states* is a *symmetry principle*, and while not every classical symmetry carries through to quantum field theory, for example, the chiral anomaly (e.g., [11], section IV.7), there is no apparent *a priori* reason to believe that whatever classical symmetries are found for these monopoles (such as (3.5)) will *only* manifest in the classical but not the quantum field theory. At the very least, the question for study becomes: do these symmetries carry over from classical to quantum field theory, and if not, why not, and in what manner are they altered? Further, if the baryon charge really is $P' = -idGG$, then as we turn $\oiint F(G) \to \oiint F\left(G\left(J(\psi)\right)\right)$, so too would we turn $P'(G) \to P'\left(G\left(J(\psi)\right)\right)$. This may reveal that the inherently-composite nature of this $P' = -idGG$ charge is in fact the long-sought "glue" to aggregate quarks and gluons together into a single *charge system*, *ab initio*.

Additionally, approaching confinement starting from a classical treatment of baryons has validating precedent in the MIT Bag Model reviewed in, e.g., [12], section 18. Irrespective of the specifics of any particular bag-type model of confinement, the MIT Bag Model very correctly makes one very important point: *focus carefully on what flows and does not flow across any closed two-dimensional surface.* And it does so using the *classical* formulation of Gauss' / Stokes' theorem. This is why the integral form of Maxwell's equations in classical field theory may well be a very sensible starting point studying confinement, because from the Bag Model viewpoint, confinement is all about what passes and does not pass through closed surfaces containing the extended field configuration within the baryon volume.

Further, by talking about the "classical level" of "non-abelian gauge theory" right on page 1 of [6], Jaffe and Witten themselves recognize that Yang-Mills theory *has a classical level*, and that a reasonable starting point for developing quantum Yang-Mills theory, is to first fully and properly develop and understand Yang-Mills gauge theory at this classical level.

Finally, it is certainly unrealistic to expect that a classical-only treatment of baryons based on Yang-Mills magnetic monopoles will explain *all* of the observed phenomenology of baryons. It cannot and will not. Only a proper quantum field treatment may be expected to do so. Yet, at the same time, there are some important physics insights to be gained even from a classical treatment of the Yang-Mills monopole equation (3.3). And we know, if we can fully develop a classical theory on its own terms, and then obtain its Lagrangian density $\mathfrak{L}(\phi)$ and action $S(\phi)$ in terms of its fields ϕ, that we can then convert over to a quantum field theory via the path integration $Z = \int D\phi \exp i \int \mathfrak{L} d^4x = \int D\phi \exp iS$. While carrying out the path integration of a non-linear theory such as Yang-Mills gauge theory (and especially gravitational theory) is still an exceptionally challenging problem, that does not mean one ought not make the effort to find the correct road for doing so, which road is revealed in section 8 and used to carry out an analytically-exact path integration in section 11. But this all this begins by finding and fleshing out, the right classical theory to quantize.

So what is most important is for researchers in particle, baryon and nuclear theory to be aware of the possibility of modelling baryons as Yang-Mills magnetic monopoles to gain possible insight into confinement and related QCD symmetries, so that this possible connection can be further developed, vetted, and empirically-tested by anyone who finds it interesting or promising. We now explore the next several steps in this development.

5. Classical Field Equations for the Yang-Mills Electric Charge

Now let us develop the electric charge density $*J$ in (1.12). Once again, via the same type of calculation used to go from (1.5) a.k.a. (1.7) to (1.11), which was also used to go from (2.5) to (2.6), together with $F = DG = dG - i[G,G]$, we write (1.12) for $*J$ in commutator form:

$$\begin{aligned} *J &= D*F = D*DG = d*F - i[G,*F] = d*(dG - i[G,G]) - i\left[G,*(dG - i[G,G])\right] \\ &= d*dG - id*[G,G] - i[G,*dG] - \left[G,*[G,G]\right] \end{aligned} . \tag{5.1}$$

This should be contrasted with the analog for P in the middle line of (2.6). Above, however, we do not have all the zeroes that were in (2.6), namely, ddG, $P=0$, and $\left[G,[G,G]\right]=0$.

As in (2.7) to (2.10), we expand the differential forms of each term. We first have:

$$*J = \tfrac{1}{3!} * J_{\sigma\mu\nu} dx^\sigma \wedge dx^\mu \wedge dx^\nu = *J_{\sigma\mu\nu} dx^\sigma dx^\mu dx^\nu , \tag{5.2}$$

$$\begin{aligned} d*dG &= \tfrac{1}{3!}\left(\partial_{;\sigma} * \partial_{;[\mu} G_{\nu]} + \partial_{;\mu} * \partial_{;[\nu} G_{\sigma]} + \partial_{;\nu} * \partial_{;[\sigma} G_{\mu]}\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2!}\partial_{;\sigma} * \partial_{;[\mu} G_{\nu]} dx^\sigma \wedge dx^\mu \wedge dx^\nu = \tfrac{1}{2!}\tfrac{1}{2!}\partial_{;\sigma}\left((-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial^{;[\alpha} G^{\beta]}\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2!}\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial_{;\sigma}\partial^{;[\alpha} G^{\beta]} dx^\sigma \wedge dx^\mu \wedge dx^\nu = \tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial_{;\sigma}\partial^{;\alpha} G^{\beta} dx^\sigma \wedge dx^\mu \wedge dx^\nu \end{aligned} . \tag{5.3}$$

Above, we have used the duality relationship $*\partial_{;[\mu} G_{\nu]} = \tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial^{;[\alpha} G^{\beta]}$. We have also allowed for a curved spacetime by using the covariant derivatives, as well as the product rule which simplifies to $\partial_{;\sigma}\left((-g)^{.5}\partial^{;[\alpha} G^{\beta]}\right) = (-g)^{.5}\partial_{;\sigma}\partial^{;[\alpha} G^{\beta]}$ because of the metricity $g_{\mu\nu;\sigma} = 0$. In flat spacetime, $\partial_{;\sigma} \to \partial_\sigma$ and $\left(-g^{.5}\right) = 1$.

Next, in contrast to (2.8), using $*\left[G_\mu, G_\nu\right] = \tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left[G^\alpha, G^\beta\right]$ and of course $g_{\mu\nu;\sigma} = 0$, with the analogous sign reversal at the sixth line as in (2.8), we have:

$$
\begin{aligned}
&-id*[G,G]=-\tfrac{1}{2!}i\partial_{;\sigma}*\left[G_\mu,G_\nu\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial_{;\sigma}\left[G^\alpha,G^\beta\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu=-\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial_{;\sigma}\left(G^\alpha G^\beta\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(\partial_{;\sigma}G^\alpha G^\beta\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu-\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(G^\alpha\partial_{;\sigma}G^\beta\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(\partial_{;\sigma}G^{[\alpha}G^{\beta]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu-\tfrac{1}{2!}\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(G^{[\alpha}\partial_{;\sigma}G^{\beta]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\left(\partial_{;\sigma}*G_{[\mu}G_{\nu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu-\tfrac{1}{2!}i\left(*G_{[\mu}\partial_{;\sigma}G_{\nu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\left(\partial_{;\sigma}*G_{[\mu}G_{\nu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu+\tfrac{1}{2!}i\left(*G_{[\sigma}\partial_{;\mu}G_{\nu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-i\left(\partial_{;\sigma}*G_\mu G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu+i\left(*G_\sigma\partial_{;\mu}G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{3!}i\left(\partial_{;\sigma}*G_{[\mu}G_{\nu]}+\partial_{;\mu}*G_{[\nu}G_{\sigma]}+\partial_{;\nu}*G_{[\sigma}G_{\mu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&\quad+\tfrac{1}{3!}i\left(*G_{[\sigma}\partial_{;\mu}G_{\nu]}+*G_{[\mu}\partial_{;\nu}G_{\sigma]}+*G_{[\nu}\partial_{;\sigma}G_{\mu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{3!}i\left(*\partial_{;[\sigma}G_{\mu]}G_\nu+*\partial_{;[\mu}G_{\nu]}G_\sigma+*\partial_{;[\nu}G_{\sigma]}G_\mu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&\quad+\tfrac{1}{3!}i\left(G_\sigma*\partial_{;[\mu}G_{\nu]}+G_\mu*\partial_{;[\nu}G_{\sigma]}+G_\nu*\partial_{;[\sigma}G_{\mu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=\left(-i*\partial_{;\sigma}G_\mu G_\nu+iG_\sigma*\partial_{;\mu}G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-i*dGG+iG*dG
\end{aligned}
\qquad ,(5.4)
$$

Note that within the differential forms, and given $*F_{\mu\nu}=\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}$ and $g_{\mu\nu;\sigma}=0$, we are able to “transfer” the duality operation, i.e., that we are able to set $\partial_{;\sigma}*G_{[\mu}G_{\nu]}\to*\partial_{;[\sigma}G_{\mu]}G_\nu$, etc. and $*G_{[\sigma}\partial_{;\mu}G_{\nu]}\to G_\sigma*\partial_{;[\mu}G_{\nu]}$, etc. This reveals $d*[G,G]=*dGG-G*dG$ as a duality product-rule identity, contrast $d[G,G]=dGG-GdG$ from (2.8).

Similarly, in contrast to (2.9), using $*\partial_{;[\mu}G_{\nu]}=\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial^{;[\alpha}G^{\beta]}$, with a sign reversal as previously in the sixth line, and transferring $*G_{[\sigma}\partial_{;\mu]}G_\nu\to G_\sigma*\partial_{;[\mu}G_{\nu]}$ in the eighth line as was done in (5.4) above without repeating the expansion to third rank tensor form, we obtain:

$$\begin{aligned}
&-i[G,*dG]=-\tfrac{1}{3!}i\left(\left[G_\sigma,*\partial_{;[\mu}G_{\nu]}\right]+\left[G_\mu,*\partial_{;[\nu}G_{\sigma]}\right]+\left[G_\nu,*\partial_{;[\sigma}G_{\mu]}\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\left[G_\sigma,*\partial_{;[\mu}G_{\nu]}\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu=-\tfrac{1}{2!}\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left[G_\sigma,\partial^{;[\alpha}G^{\beta]}\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left[G_\sigma,\partial^{;\alpha}G^\beta\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(G_\sigma\partial^{;\alpha}G^\beta-\partial^{;\alpha}\left(G^\beta G_\sigma\right)\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(G_\sigma\partial^{;\alpha}G^\beta-G^\beta\partial^{;\alpha}G_\sigma-\partial^{;\alpha}G^\beta G_\sigma\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}\tfrac{1}{2!}i(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left(G_\sigma\partial^{;[\alpha}G^{\beta]}+G^{[\alpha}\partial^{;\beta]}G_\sigma-\partial^{;[\alpha}G^{\beta]}G_\sigma\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\left(G_\sigma*\partial_{;[\mu}G_{\nu]}+*G_{[\sigma}\partial_{;\mu]}G_\nu-*\partial_{;[\sigma}G_{\mu]}G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\left(2G_\sigma*\partial_{;[\mu}G_{\nu]}-*\partial_{;[\sigma}G_{\mu]}G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=\left(-2iG_\sigma*\partial_{;\mu}G_\nu+i*\partial_{;\sigma}G_\mu G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-2iG*dG+i*dGG
\end{aligned}\quad. \qquad (5.5)$$

Finally, in contrast to (2.10), using $*\left[G_\mu,G_\nu\right]=\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left[G^\alpha,G^\beta\right]$,

$$\begin{aligned}
&-\left[G,*[G,G]\right]=-\tfrac{1}{3!}\left(\left[G_\sigma,*\left[G_\mu,G_\nu\right]\right]+\left[G_\mu,*\left[G_\nu,G_\sigma\right]\right]+\left[G_\nu,*\left[G_\sigma,G_\mu\right]\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}\left[G_\sigma,*\left[G_\mu,G_\nu\right]\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu=-\tfrac{1}{2!}\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left[G_\sigma,\left[G^\alpha,G^\beta\right]\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}\tfrac{1}{3!}(-g)^{.5}\left(\varepsilon_{\alpha\beta\mu\nu}\left[G_\sigma,\left[G^\alpha,G^\beta\right]\right]+\varepsilon_{\alpha\beta\nu\sigma}\left[G_\mu,\left[G^\alpha,G^\beta\right]\right]+\varepsilon_{\alpha\beta\sigma\mu}\left[G_\nu,\left[G^\alpha,G^\beta\right]\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\neq 0
\end{aligned}\quad .(5.6)$$

Unlike (2.10), this is *not* zero via the Jacobian identity $\left[a,[b,c]\right]+\left[b,[c,a]\right]+\left[c,[a,b]\right]=0$, because although $\left[G^\alpha,G^\beta\right]$ is common to each of the three terms in the bottom line of (5.6), $\varepsilon_{\alpha\beta\mu\nu}G_\sigma\neq\varepsilon_{\alpha\beta\nu\sigma}G_\mu\neq\varepsilon_{\alpha\beta\sigma\mu}G_\nu$ are three distinct tensors.

So now we use $-id*[G,G]=-i*dGG+iG*dG$ and $-i[G,*dG]=-2iG*dG+i*dGG$ found in (5.4) and (5.5), in (5.1). Analogously to (2.11) we obtain:

$$\begin{aligned}
*J&=d*dG-id*[G,G]-i[G,*dG]-\left[G,*[G,G]\right]\\
&=d*dG-i*dGG+iG*dG-2iG*dG+i*dGG-\left[G,*[G,G]\right]\\
&=d*dG-iG*dG-\left[G,*[G,G]\right]\\
&=d*dG+id*[G,G]-i*dGG-\left[G,*[G,G]\right]\\
&=d*F-i[G,*dG]-\left[G,*[G,G]\right]\\
&=d*F-2iG*dG+i*dGG-\left[G,*[G,G]\right]
\end{aligned}\quad. \qquad (5.7)$$

This corresponds to (2.11), however, here, a) $*J \neq 0$ in contrast to $P=0$; b) $d*dG \neq 0$ in contrast to $ddG=0$; c) $\left[G,*[G,G]\right] \neq 0$ in contrast to $\left[G,[G,G]\right]=0$, and d) the terms $id[G,G] \rightarrow id*[G,G]$ and $-idGG \rightarrow -i*dGG$. Based on the top line, we also use $*F = *\left(dG - i[G,G]\right)$ which is the differential form for $*F_{\mu\nu} = *\left(\partial_{[\mu}G_{\nu]} - i\left[G_\mu, G_\nu\right]\right)$ in the final two lines.

Now we wish to apply Gauss' / Stokes' theorem to (5.7), as we earlier did to (2.11). Using the last two lines of (5.7) with the integrable term $d*F$ separated on the left, we have:

$$\begin{aligned}\oint\!\!\oint *F &= \iiint d*F \\ &= \iiint \left(*J + i[G,*dG] + \left[G,*[G,G]\right]\right) \\ &= \iiint \left(*J - i*dGG + 2iG*dG + \left[G,*[G,G]\right]\right)\end{aligned} \quad . \tag{5.8}$$

The Abelian portion of this equation, $\oint\!\!\oint *F = \iiint *J$ which we used for pedagogic simplicity in the analysis following (3.3), is clearly included when the gauge fields are set to zero. Putting the Yang-Mills electric charge equation (5.8) together with the magnetic charge equation (3.3), we find that Maxwell's Yang-Mills equations in integral form are:

$$\begin{aligned}\oint\!\!\oint *F &= -i\iiint *dGG + \iiint *J + \iiint \left(2iG*dG + \left[G,*[G,G]\right]\right) \\ \oint\!\!\oint F &= -i\iiint dGG = -i\oint\!\!\oint [G,G]\end{aligned} \quad . \tag{5.9}$$

In this form, the parallels and differences are manifestly clear. $\oint\!\!\oint *F$ is the net electric field flux and $\oint\!\!\oint F$ the net magnetic field flux over a closed surface. $*J$ is the electric source charge density and it is non-vanishing, while the magnetic source density $P=0$ vanishes by the Jacobian (2.4). Similarly, while $G*dG \neq 0$ and $\left[G,*[G,G]\right] \neq 0$ in the electric field equation, their duality counterparts $GdG=0$ and $\left[G,[G,G]\right]=0$ are also part of the magnetic charge equation, but vanish by the respective identities found in (2.11) and (2.10). We see how the only true, elementary source is $*J$ and that there are then a number of faux sources which include $P' = -idGG$ for the net magnetic field flux $\oint\!\!\oint F$, and $*J' \equiv -i*dGG + 2iG*dG + \left[G,*[G,G]\right]$ which is a faux electric source which contributes to the net electric field flux beyond that contributed by "true" electric source J in the abelian portion $\oint\!\!\oint *F = \iiint *J$ of (5.9).

Because the only elementary, real, not-faux source in the Yang-Mills equations (5.9) is the electric source $*J$, it will be desirable to solve the electric charge density equation (5.7) for the gauge field G in terms of $*J$. Particularly, as laid out at the end of section 3, our eventual goal is to find $\oint\!\!\oint F\left(G\left(J(\psi)\right)\right)$. So a key step along the way is to obtain the gauge fields $G(J)$

in terms of sources. Equation (5.7) has a number of alternative ways to express $*J(G)$, but the most compact way is on the third line. So we expand those differential forms to obtain:

$$\begin{aligned}
&*J=\tfrac{1}{3!}*J_{\sigma\mu\nu}dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\
&=d*dG-iG*dG-\left[G,*\left[G,G\right]\right]\\
&=\tfrac{1}{3!}\left(\partial_{\sigma}*\partial_{[\mu}G_{\nu]}+\partial_{\mu}*\partial_{[\nu}G_{\sigma]}+\partial_{\nu}*\partial_{[\sigma}G_{\mu]}\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\
&-\tfrac{1}{3!}i\left(G_{\sigma}*\partial_{[\mu}G_{\nu]}+G_{\mu}*\partial_{[\nu}G_{\sigma]}+G_{\nu}*\partial_{[\sigma}G_{\mu]}\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\
&-\tfrac{1}{3!}\left(\left[G_{\sigma},*\left[G_{\mu},G_{\nu}\right]\right]+\left[G_{\mu},*\left[G_{\nu},G_{\sigma}\right]\right]+\left[G_{\nu},*\left[G_{\sigma},G_{\mu}\right]\right]\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}
\end{aligned} \quad . \tag{5.10}$$

Stripping off the forms, we obtain the tensor equation:

$$\begin{aligned}
J_{\sigma\mu\nu}=&\left(\partial_{\sigma}\partial_{[\mu}G_{\nu]}+\partial_{\mu}*\partial_{[\nu}G_{\sigma]}+\partial_{\nu}*\partial_{[\sigma}G_{\mu]}\right)\\
&-i\left(G_{\sigma}*\partial_{[\mu}G_{\nu]}+G_{\mu}*\partial_{[\nu}G_{\sigma]}+G_{\nu}*\partial_{[\sigma}G_{\mu]}\right)\\
&-\left(\left[G_{\sigma},*\left[G_{\mu},G_{\nu}\right]\right]+\left[G_{\mu},*\left[G_{\nu},G_{\sigma}\right]\right]+\left[G_{\nu},*\left[G_{\sigma},G_{\mu}\right]\right]\right)
\end{aligned} \quad . \tag{5.11}$$

Then, we apply the duality operations $*J_{\sigma\mu\nu}=(-g)^{.5}\varepsilon_{\alpha\sigma\mu\nu}J^{\alpha}$, $*\partial_{[\mu}G_{\nu]}=\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\partial^{[\alpha}G^{\beta]}$ and $*\left[G_{\mu},G_{\nu}\right]=\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}\left[G^{\alpha},G^{\beta}\right]$, and the metricity $g_{\mu\nu;\sigma}=0$ as discussed after (5.3), to obtain (a good summary of the use of duality is contained in [9], pages 87-89):

$$\begin{aligned}
&(-g)^{.5}\varepsilon_{\alpha\sigma\mu\nu}J^{\alpha}\\
&=\tfrac{1}{2!}(-g)^{.5}\left(\varepsilon_{\alpha\beta\mu\nu}\partial_{\sigma}\partial^{[\alpha}G^{\beta]}+\varepsilon_{\alpha\beta\nu\sigma}\partial_{\mu}\partial^{[\alpha}G^{\beta]}+\varepsilon_{\alpha\beta\sigma\mu}\partial_{\nu}\partial^{[\alpha}G^{\beta]}\right)\\
&-\tfrac{1}{2!}i(-g)^{.5}\left(\varepsilon_{\alpha\beta\mu\nu}G_{\sigma}\partial^{[\alpha}G^{\beta]}+\varepsilon_{\alpha\beta\nu\sigma}G_{\mu}\partial^{[\alpha}G^{\beta]}+\varepsilon_{\alpha\beta\sigma\mu}G_{\nu}\partial^{[\alpha}G^{\beta]}\right)\\
&-\tfrac{1}{2!}(-g)^{.5}\left(\varepsilon_{\alpha\beta\mu\nu}\left[G_{\sigma},\left[G^{\alpha},G^{\beta}\right]\right]+\varepsilon_{\alpha\beta\nu\sigma}\left[G_{\mu},\left[G^{\alpha},G^{\beta}\right]\right]+\varepsilon_{\alpha\beta\sigma\mu}\left[G_{\nu},\left[G^{\alpha},G^{\beta}\right]\right]\right)
\end{aligned} \quad . \tag{5.12}$$

Factoring out $(-g)^{.5}$ and multiplying through by $\varepsilon^{\kappa\sigma\mu\nu}$ next yields:

$$
\begin{aligned}
&\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\sigma\mu\nu}J^{\alpha}=-3!\delta^{\kappa}{}_{\alpha}J^{\alpha}=-6J^{\kappa}\\
&=\tfrac{1}{2!}\left(\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\mu\nu}\partial_{\sigma}\partial^{[\alpha}G^{\beta]}+\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\nu\sigma}\partial_{\mu}\partial^{[\alpha}G^{\beta]}+\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\sigma\mu}\partial_{\nu}\partial^{[\alpha}G^{\beta]}\right)\\
&-\tfrac{1}{2!}i\left(\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\mu\nu}G_{\sigma}\partial^{[\alpha}G^{\beta]}+\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\nu\sigma}G_{\mu}\partial^{[\alpha}G^{\beta]}+\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\sigma\mu}G_{\nu}\partial^{[\alpha}G^{\beta]}\right)\\
&-\tfrac{1}{2!}\left(\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\mu\nu}\left[G_{\sigma},\left[G^{\alpha},G^{\beta}\right]\right]+\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\nu\sigma}\left[G_{\mu},\left[G^{\alpha},G^{\beta}\right]\right]+\varepsilon^{\kappa\sigma\mu\nu}\varepsilon_{\alpha\beta\sigma\mu}\left[G_{\nu},\left[G^{\alpha},G^{\beta}\right]\right]\right).\\
&=-\left(\delta^{\kappa\sigma}{}_{\alpha\beta}\partial_{\sigma}\partial^{[\alpha}G^{\beta]}+\delta^{\kappa\mu}{}_{\alpha\beta}\partial_{\mu}\partial^{[\alpha}G^{\beta]}+\delta^{\kappa\nu}{}_{\alpha\beta}\partial_{\nu}\partial^{[\alpha}G^{\beta]}\right)\\
&\quad+i\left(\delta^{\kappa\sigma}{}_{\alpha\beta}G_{\sigma}\partial^{[\alpha}G^{\beta]}+\delta^{\kappa\mu}{}_{\alpha\beta}G_{\mu}\partial^{[\alpha}G^{\beta]}+\delta^{\kappa\nu}{}_{\alpha\beta}G_{\nu}\partial^{[\alpha}G^{\beta]}\right)\\
&\quad+\left(\delta^{\kappa\sigma}{}_{\alpha\beta}\left[G_{\sigma},\left[G^{\alpha},G^{\beta}\right]\right]+\delta^{\kappa\mu}{}_{\alpha\beta}\left[G_{\mu},\left[G^{\alpha},G^{\beta}\right]\right]+\delta^{\kappa\nu}{}_{\alpha\beta}\left[G_{\nu},\left[G^{\alpha},G^{\beta}\right]\right]\right)
\end{aligned}
\tag{5.13}
$$

Using $\delta^{\kappa\sigma}{}_{\alpha\beta}\equiv\delta^{\kappa}{}_{\alpha}\delta^{\sigma}{}_{\beta}-\delta^{\kappa}{}_{\beta}\delta^{\sigma}{}_{\alpha}$ and the like, with $\kappa\to\nu$ index renaming, this reduces to:

$$
-J^{\nu}=\partial_{\sigma}\partial^{[\sigma}G^{\nu]}-iG_{\sigma}\partial^{[\sigma}G^{\nu]}-\left[G_{\sigma},\left[G^{\sigma},G^{\nu}\right]\right].
\tag{5.14}
$$

Contrasting to the original $*J=d*dG-iG*dG-\left[G,*\left[G,G\right]\right]$, we see that aside from the sign reversal, the * between two objects essentially results in an index contraction between those two objects when they are written as tensors. If we then expand all the commutators and reorganize terms in a familiar way, we obtain:

$$
\begin{aligned}
-J^{\nu}&=\partial_{\sigma}\partial^{[\sigma}G^{\nu]}-iG_{\sigma}\partial^{[\sigma}G^{\nu]}-\left[G_{\sigma},\left[G^{\sigma},G^{\nu}\right]\right]\\
&=\left(\partial_{\sigma}\partial^{\sigma}-iG_{\sigma}\partial^{\sigma}-G_{\sigma}G^{\sigma}\right)G^{\nu}-\left(\partial^{\sigma}\partial^{\nu}-iG^{\sigma}\partial^{\nu}-2G^{\sigma}G^{\nu}+G^{\nu}G^{\sigma}\right)G_{\sigma}\\
&=g^{\nu\sigma}\left(\partial_{\tau}\partial^{\tau}-iG_{\tau}\partial^{\tau}-G_{\tau}G^{\tau}\right)G_{\sigma}-\left(\partial^{\sigma}\partial^{\nu}-iG^{\sigma}\partial^{\nu}-2G^{\sigma}G^{\nu}+G^{\nu}G^{\sigma}\right)G_{\sigma}\\
&\equiv\left(g^{\nu\sigma}D_{\tau}D^{\tau}-D^{\sigma}D^{\nu}\right)G_{\sigma}
\end{aligned},
\tag{5.15}
$$

with a configuration space operator $g^{\nu\sigma}D_{\tau}D^{\tau}-D^{\sigma}D^{\nu}$ where in the final line we have defined:

$$
D^{\sigma}D^{\nu}\equiv\partial^{\sigma}\partial^{\nu}-iG^{\sigma}\partial^{\nu}-2G^{\sigma}G^{\nu}+G^{\nu}G^{\sigma}
\tag{5.16}
$$

which, upon contraction, does yield:

$$
D_{\tau}D^{\tau}=\partial_{\tau}\partial^{\tau}-iG_{\tau}\partial^{\tau}-G_{\tau}G^{\tau}.
\tag{5.17}
$$

By way of contrast, in Abelian gauge theory, $-J^{\nu}=\left(g^{\nu\sigma}\partial_{\tau}\partial^{\tau}-\partial^{\sigma}\partial^{\nu}\right)G_{\sigma}$. So (5.15) for $J^{\nu}\left(G_{\sigma}\right)$, is now in a familiar form which we can use to approach taking the inverse $G_{\sigma}\left(J^{\nu}\right)$. This is the first step toward being able to obtain $\oiint F\left(G\left(J\left(\psi\right)\right)\right)$.

Finally, let us find the continuity equation for conservation of the electric source density and current, based on (5.15). Equation (5.15) will clearly be recognized as another way to express $-J^{\nu}=D_{\sigma}F^{\sigma\nu}$ which may be similarly derived from $*J=D*F$ in (1.12). Particularly, we wish to show that $-D_{\nu}J^{\nu}=D_{\nu}D_{\sigma}F^{\sigma\nu}=0$, by identity. Similarly to (2.1), we may take the gauge-covariant derivative of J^{ν} via the commutation:

$$\begin{aligned}&\left[D_{\nu},J^{\nu}\right]\varphi=D_{\nu}\left(J^{\nu}\varphi\right)-J^{\nu}D_{\nu}\varphi=\left(\partial_{\nu}-iG_{\nu}\right)\left(J^{\nu}\varphi\right)-J^{\nu}\left(\partial_{\nu}-iG_{\nu}\right)\varphi\\&=\partial_{\nu}J^{\nu}\varphi+J^{\nu}\partial_{\nu}\varphi-iG_{\nu}J^{\nu}\varphi-J^{\nu}\partial_{\nu}\varphi+iJ^{\nu}G_{\nu}\varphi=\partial_{\nu}J^{\nu}\varphi-i\left[G_{\nu},J^{\nu}\right]\varphi=D_{\nu}J^{\nu}\varphi\end{aligned}. \tag{5.18}$$

Stripping off the φ, we see the correct derivative:

$$\left[D_{\nu},J^{\nu}\right]=\partial_{\nu}J^{\nu}-i\left[G_{\nu},J^{\nu}\right]=D_{\nu}J^{\nu} \tag{5.19}$$

which includes the commutator $\left[G_{\nu},J^{\nu}\right]$. So, we start with $-D_{\nu}J^{\nu}=D_{\nu}D_{\sigma}F^{\sigma\nu}$ and apply $\left[D_{\sigma},F_{\mu\nu}\right]=D_{\sigma}F_{\mu\nu}$ from (2.2), $\left[D_{\nu},J^{\nu}\right]=D_{\nu}J^{\nu}$ from (5.19), $-J^{\nu}=D_{\sigma}F^{\sigma\nu}$, and $iF_{\sigma\nu}=\left[D_{\nu},D_{\sigma}\right]$ from (1.1) to show via simple index commutativity that the continuity equation, due to the scalar contraction $F_{\sigma\nu}F^{\sigma\nu}$ of like-objects, is:

$$\begin{aligned}&-D_{\nu}J^{\nu}=-\left[D_{\nu},J^{\nu}\right]=\left[D_{\nu},D_{\sigma}F^{\sigma\nu}\right]=\left[D_{\nu},\left[D_{\sigma},F^{\sigma\nu}\right]\right]\\&=D_{\nu}D_{\sigma}F^{\sigma\nu}-D_{\nu}F^{\sigma\nu}D_{\sigma}-D_{\sigma}F^{\sigma\nu}D_{\nu}+F^{\sigma\nu}D_{\sigma}D_{\nu}\\&=D_{\nu}D_{\sigma}F^{\sigma\nu}+F^{\sigma\nu}D_{\sigma}D_{\nu}=\left[D_{\nu}D_{\sigma},F^{\sigma\nu}\right]=\tfrac{1}{2}\left[\left[D_{\nu},D_{\sigma}\right],F^{\sigma\nu}\right]\\&=\tfrac{1}{2}i\left[F_{\sigma\nu},F^{\sigma\nu}\right]=0\end{aligned}. \tag{5.20}$$

The continuity equation in differential forms, therefore, is $D*J=DD*F=0$. This equation for the conservation of the non-abelian charge density will play a very central role the development to follow.

6. Abelian and non-Abelian Massive Gauge Boson Inverses for the Electric Charge Density, Using the "Coleman-Zee" Method

The next stage in our development to demonstrate that $\oiint F=-i\iiint dGG$ in (5.9) is the integral-form classical equation for a baryon, is to invert the configuration space operator $g^{\nu\sigma}D_{\tau}D^{\tau}-D^{\sigma}D^{\nu}$ of (5.15) to obtain $G_{\sigma}\left(J^{\nu}\right)$, so we can obtain $\oiint F\left(G\left(J\right)\right)$. This inverse, which we denote by $I_{\mu\nu}$, may be defined by $G_{\mu}\equiv I_{\mu\nu}J^{\nu}$. In general, $I_{\mu\nu}\neq I_{\nu\mu}$ is not necessarily symmetric, so $G_{\mu}\equiv I_{\mu\nu}J^{\nu}$ is an inner product definition not necessarily the same as an outer product definition $G_\mu \equiv I'_{\nu\mu} J^\nu$. Making use of $G_\mu \equiv I_{\mu\nu} J^\nu$ to left-multiply (5.15) by $-I_{\mu\nu}$ allows us to write:

$$I_{\mu\nu} J^\nu = -I_{\mu\nu}\left(g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu\right) G_\sigma = G_\mu = \delta^\sigma{}_\mu G_\sigma , \tag{6.1}$$

from which we may extract a more-directly defined inverse:

$$-I_{\mu\nu}\left(g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu\right) = \delta^\sigma{}_\mu . \tag{6.2}$$

Now the task is to show that this inverse exists, to understand the degree to which any particular inverse which does exist is non-unique, to review the options for fixing the gauge of these inverses, and to select the inverse or inverses with suitable gauge choices or better yet, *unique gauge requirements* which best illustrate why $\oiint F = -i \iiint dGG$ based on a faux magnetic charge $P' = -idGG$ of (3.4) has all of the key symmetries of a baryon.

Taking inverses in gauge theory is a tricky business, because one is often free to choose the gauge resulting in non-unique inverses, and because particularly for massless gauge bosons – which include the gluons of QCD – the inverse *may not even exist* without a careful selection and fixing of the gauge, see, e.g., [11] chapter III.4. Additionally, because the gauge field is the field of integration used to turn a classical action *S* into a quantum field amplitude *W*, a symmetry that exists classically may not be a symmetry of the related quantum field theory, see, e.g., [11] chapter IV.7 (Chiral Anomaly). Specifically, a classical symmetry exists if some transformation leaves the action $S(\varphi)$ invariant. A quantum symmetry exists if (and inherits the classical symmetry) if the same transformation leaves the path integral $Z = \int D\varphi \exp iS(\varphi)$ invariant. But this may not always be the case. Therefore, let us start by carefully parsing out the various issues that come into play when taking inverses of the form (6.2).

First, as to classical versus quantum fields, we consider the local non-abelian gauge transformation which is $G_\mu \to G'_\mu = G_\mu + \partial_\mu \theta - i\left[G_\mu, \theta\right]$ in tensors, $G \to G' = G + d\theta - i[G, \theta]$ in differential commutator forms, and $G \to G' = G + d\theta + G \wedge \theta = G + (d + G\wedge)\theta$ in differential wedge forms. These are all alternative but equivalent ways of saying the same thing. All of the classical field equations developed thus far including (1.12), (2.11), (3.3), (5.1), (5.7) and (5.9) are symmetric under such a gauge transformation. So too, the electric charge field equation (5.15) with the specific $D^\sigma D^\nu$ and $D_\tau D^\tau$ identified in (5.16) and (5.17) is symmetric under this non-abelian gauge transformation. This should be no surprise: all of these equations were developed with the express purpose of preserving this gauge symmetry. This means that the action $S(G) = \int \mathfrak{L}(G) d^4x$ is similarly invariant. But when we take a path integral $Z = \int DG \exp iS(G) \equiv \mathcal{C} \exp iW(J)$ to obtain the associated quantum field theory for the amplitude $W(J)$, we see that we are not necessarily assured that the measure DG will have this same symmetry. And this in turn means that the quantum field theory may not share all of the symmetries of the classical field theory. Typically, ensuring that the path integral also carries forward the gauge symmetry under $DG \to D(G + d\theta - i[G,\theta])$ is what gives rise to gauge-fixing measures such as Faddeev-Popov [13] including anticommuting scalar "ghost" fields, see some concise development of this in [11], chapters III.4, and VII.1. However, so long as we restrict ourselves to classical field theory, which we are doing at the moment, we can develop inverses without this particular worry. We just need to be prepared to address this issue once we are ready to calculate the path integral, which is to be done only after the classical theory has been fully elaborated. Again, as to why there is both validity and benefit to doing taking this approach of fully elaborating the classical theory in advance of the quantum theory, see the discussion of section 4.

Second, as to why we need to take inverses when going from classical to quantum field theory, this is because the mathematical exercise of calculating a path integral revolves around clever extrapolations of the Gaussian integral $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right) = \left(-2\pi / A\right)^{.5} \exp\left(J^2 / 2A\right)$ into $Z = \int DG \exp\left(iS(G)\right) \equiv C \exp\left(iW(J)\right)$, with the correspondence $W(J) \sim J^2 / 2A$. Because the abstracted coefficient A of Ax^2 gets inverted in $J^2 / 2A$, and because A ends up corresponding with the configuration space operator $g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu$ in (6.2) which then gets inverted via $J^2 / 2A$ into $I_{\nu\mu}$ which then becomes proportionately related to the quantum propagator assuming we can find a way as we will in sections 8 and 11 to deal with $g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu$ *not* being quadratic in G_μ, one must expect to have to obtain $\left(g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu\right)^{-1}$ to arrive at quantum field theory, in addition to having to deal with the invariance of the measure under $DG \to D(G + d\theta - i[G,\theta])$. Thus, it is desirable to have a number of inverses already developed "on the shelf" when it comes time to use them to calculate a path integral. But, as we see in (6.2), even before we start approaching path integration, we still need this inverse *even to develop the classical theory*, and specifically, in order to obtain $\oiint F(G(J))$.

Third, even in classical theory, configuration space operators of the form $g^{\nu\sigma}\partial_\tau\partial^\tau - \partial^\sigma\partial^\nu$ simply have no inverse! Although often couched in mystery, this problem arises from the simple fact that for a massless gauge boson, a Lorentz vector G_μ with *four* spacetime components is used to describe physical fields – for example the photon in electrodynamics and the gluons in chromodynamics – which only have *two* physical degrees of freedom. That is, a *mathematical object* G_μ with four degrees of freedom is used to represent a *physical object* which only has half as many degrees of freedom. This is an inherent redundancy in how we describe gauge fields that causes inverses to be non-unique and brings about the need for gauge fixing. Gauge fixing and related methods are then used to create a menu of gauge-fixed solutions out of the non-uniqueness stemming from this redundancy. *This gauge non-uniqueness is a separate and distinct issue from gauge symmetry*. For example, the field equation $-J^\nu = \left(g^{\nu\sigma}\partial_\tau\partial^\tau - \partial^\sigma\partial^\nu\right) A_\sigma$ for a photon field A_σ sourced by a current density J^ν is fully symmetric under an abelian gauge transformation $A_\sigma \to A'_\sigma = A_\sigma + \partial_\sigma \theta$. But A_σ is still redundant insofar as it has four spacetime degrees of freedom while a photon only has two transverse degrees of freedom. Additionally, as mentioned, the operator $g^{\nu\sigma}\partial_\tau \partial^\tau - \partial^\sigma \partial^\nu$ has no inverse, or, to be more precise, has an inverse which is of infinite magnitude and so is completely indeterminate.

Now, as Zee points out on page 30 of [11]:

> "In order to avoid complications at this stage associated with gauge invariance [we] will consider instead the field theory of a massive spin 1 meson, or vector meson. . . . We can adopt a pragmatic attitude: Calculate a photon mass *m* and set $m = 0$ at the end, and if the result does not blow up in our faces, we will presume that it is OK."

Zee states in a footnote to this passage that when he "took a field theory course as a student with Sidney Coleman this was how he treated QED to avoid discussing gauge invariance." So to simplify the development here, we shall take this same pragmatic approach as Coleman and Zee: We shall introduce a non-zero "Proca mass" for the gauge fields *G*, develop the classical monopole $\oiint F = -i \iiint dGG$ of (5.9) to show how it has all of the classical symmetries that one would expect of a baryon, and then set $m = 0$ at the appropriate point in the development (which will come at (9.15) infra) and explore the massive / massless correspondences.

In this section, we shall develop the inverse of the *massive* boson configuration space operators $g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu$ for non-abelian gauge theory and $g^{\nu\sigma}\left(\partial_\tau \partial^\tau + m^2\right) - \partial^\sigma \partial^\nu$ for abelian gauge theory, and then follow Coleman and Zee by setting the mass to zero to see what results. In the next section we will take the more formal approach of developing the inverses $g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu$ and $g^{\nu\sigma}\partial_\tau \partial^\tau - \partial^\sigma \partial^\nu$ for a *massless* particle directly, using the Faddeev-Popov method. We will then contrast the both approaches to see where they meet, to give us some guidance about how to then use these inverses in the non-abelian magnetic monopole field equation $\oiint F = -i \iiint dGG$.

So, following the Coleman-Zee approach, let us add a Proca mass *m* to (5.15), thus:

$$-J^\nu = \left(g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu\right) G_\sigma . \tag{6.3}$$

Let us then consider (6.3) in flat spacetime where gradient operators $\left[\partial_\mu, \partial_\nu\right] = 0$ commute. Let us also momentarily revert $D \to \partial$ to ordinary derivatives to make a pedagogical point, and so write (6.3) as its abelian subset $-J^\nu = \left(g^{\nu\sigma}\left(\partial_\tau \partial^\tau + m^2\right) - \partial^\sigma \partial^\nu\right) G_\sigma$. The current density is conserved by the continuity equation $\partial_\nu J^\nu = 0$, so if we take the gradient of each side and reduce, we find that $m^2 \partial_\nu G^\nu = 0$. Because we take the mass to be non-zero, this means that $\partial_\nu G^\nu = 0$, which is a fully-covariant equation known as the Lorenz gauge. Here, $\partial_\nu G^\nu = 0$ is not a gauge condition at all; it is a *requirement* needed to ensure continuity for a massive vector boson. The number of degrees of freedom in the *mathematical* object G^ν is covariantly reduced from four to three by $\partial_\nu G^\nu = 0$, and this matches precisely with the three polarization degrees of freedom – one longitudinal and two transverse – possessed by the *physical* gauge boson. So now, most of the gauge redundancy is squeezed out from G^ν. Even here, however, there is still a residual redundancy that requires gauge fixing. For, if we transform $G^\nu \to G^\nu + \partial^\nu\theta$, then the Lorenz condition becomes $\partial_\nu\left(G^\nu + \partial^\nu\theta\right) = 0$, or $\partial_\nu G^\nu = -\partial_\nu\partial^\nu\theta$. So to maintain $\partial_\nu G^\nu = 0$ under any such gauge transformation, we may this fix the gauge completely by the gauge condition $\partial_\nu\partial^\nu\theta = 0$. Therefore, with everything taken together, (6.3) is invariant under a gauge transformation $G^\nu \to G^\nu + \partial^\nu\theta$, the four degrees of freedom in G^ν are covariantly-reduced down to three degrees of freedom by $\partial_\nu G^\nu = 0$ which is required to match the three polarization degrees of freedom of the physical field, and the residual gauge freedom is fixed and thereby removed by $\partial_\nu\partial^\nu\theta = 0$. The field equation $-J^\nu = \left(g^{\nu\sigma}\left(\partial_\tau\partial^\tau + m^2\right) - \partial^\sigma\partial^\nu\right)G_\sigma$ remains invariant under the gauge transformation $G^\nu \to G^\nu + \partial^\nu\theta$ and this invariance does not depend in any way on $\partial_\nu\partial^\nu\theta = 0$ because nowhere does the non-observable gauge (really, phase) angle θ appear in the field equation.

In the non-abelian (6.3) it is a bit more complicated, because we have D from (5.15), (5.16) not ∂, and because the proper way to take the gauge-derivative of the current density is by $\left[D_\nu, J^\nu\right] = \partial_\nu J^\nu - i\left[G_\nu, J^\nu\right] = D_\nu J^\nu$ derived in (5.19). But we already saw that the continuity equation $D_\nu J^\nu = 0$ of (5.20) which we now combine with (5.15), by identity, is:

$$-D_\nu J^\nu = D_\nu\left(g^{\nu\sigma}D_\tau D^\tau - D^\sigma D^\nu\right)G_\sigma = \mathbf{0}. \tag{6.4}$$

So if we simply add a Proca mass to (6.4) and maintain continuity, we must have:

$$\begin{aligned}-D_\nu J^\nu &= D_\nu\left(g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu\right)G_\sigma = D_\nu\left(g^{\nu\sigma}D_\tau D^\tau - D^\sigma D^\nu\right)G_\sigma + m^2 D_\nu g^{\nu\sigma}G_\sigma = 0 \\ &= \mathbf{0} + m^2 D_\nu G^\nu = 0\end{aligned}. \tag{6.5}$$

This includes $D_\nu\left(g^{\nu\sigma}m^2G_\sigma\right) = D_\nu\left(m^2G^\nu\right) = m^2D_\nu G^\nu = 0$, where the highlighted zero in (6.4) and (6.5) is the zero-by-identity of the continuity equation (5.20). But the symmetries of the term $D_\nu G^\nu$ in the above are driven by those of (5.19) which is $D_\nu J^\nu = \partial_\nu J^\nu - i\left[G_\nu, J^\nu\right]$. Consequently, $D_\nu G^\nu = \partial_\nu G^\nu - i\left[G_\nu, G^\nu\right]$ because of (5.19). Additionally, because of (6.5), $D_\nu G^\nu = \partial_\nu G^\nu - i\left[G_\nu, G^\nu\right] = 0$. As in the abelian case just discussed, for a massive gauge boson, and this is not a mere gauge condition. It is *required* to ensure continuity. As in abelian theory this reduces the gauge freedom of a four-component spacetime object G_ν down to three to match the three massive boson polarizations. Additionally, here the commutator $\left[G_\nu, G^\nu\right] = 0$ because of the scalar contraction $G_\nu G^\nu$ of like objects. This means in turn that $D_\nu G^\nu = \partial_\nu G^\nu = 0$. And this means that $\partial_\nu G^\nu = 0$ *still applies even to the non-abelian theory* and is not a gauge condition but is a requirement for a massive gauge boson.

As to the residual gauge freedom, because $G^\nu \to G'^\nu = G^\nu + \partial^\nu\theta - i\left[G^\nu,\theta\right] = G^\nu + D^\nu\theta$ is the non-abelian gauge transformation, $D_\nu G'^\nu = D_\nu G^\nu + D_\nu D^\nu\theta = \partial_\nu G^\nu + \partial_\nu D^\nu\theta = 0$ is the required covariant gauge condition for G'^ν. Taken with $D_\nu G^\nu = 0$ this means that for a non-abelian theory, $D_\nu D^\nu\theta = 0$ replaces $\partial_\nu\partial^\nu\theta = 0$ as the residual gauge condition. Taken with $\partial_\nu G^\nu = 0$, this means that $\partial_\nu D^\nu\theta = \partial_\nu\partial^\nu\theta - i\partial_\nu\left[G^\nu,\theta\right] = 0$, which means that $D_\nu D^\nu\theta = 0$ may be written out with ordinary derivatives as $\partial_\nu\partial^\nu\theta - i\partial_\nu\left[G^\nu,\theta\right] = 0$. So while (6.3) is invariant under a non-abelian gauge transformation $G^\nu \to G'^\nu = G^\nu + \partial^\nu\theta - i\left[G^\nu,\theta\right]$, we are *required* to have $D_\nu G^\nu = \partial_\nu G^\nu = 0$ because the boson in (6.3) is presumed to be massive and subject to continuity, and the remaining gauge freedom is fixed by imposing $D_\nu D^\nu\theta = 0$ which as just seen is equivalent to the expression $\partial_\nu\partial^\nu\theta - i\partial_\nu\left[G^\nu,\theta\right] = 0$. Nonetheless, as in the abelian theory, this invariance does not depend in any way on $D_\nu D^\nu\theta = 0$ a.k.a. $\partial_\nu\partial^\nu\theta - i\partial_\nu\left[G^\nu,\theta\right] = 0$ because nowhere does the non-observable gauge / phase angle θ appear in the field equation (6.3).

Now, let us stop for a moment to take a close look at the gauge-covariant, second-rank, second-derivative operator $D^\sigma D^\nu$ in (5.16) and its gauge-covariant d'Alembertian $\square = D_\tau D^\tau$ of (5.17). Close study of $D^\sigma D^\nu$ will reveal that there is no apparent way to separate each of D^σ and D^ν to make $D^\sigma D^\nu$ a product of two separate expressions for D^σ, D^ν. Even the commutator of (5.16), which we can calculate to be $i\left[D^\sigma, D^\nu\right] = G^{[\sigma}\partial^{\nu]} - 3i\left[G^\sigma, G^\nu\right]$ in flat spacetime, is different from $F_{\mu\nu}\varphi = i\left[D_\mu, D_\nu\right]\varphi = \left(\partial_{[\mu}G_{\nu]} - i\left[G_\mu, G_\nu\right]\right)\varphi$ which is the field strength defined in (1.1), (1.5). This is because in (5.15) $D^\sigma D^\nu$ is operating on G_σ not φ and because, as noted at the outset following (1.1), gauge-covariant derivatives, like covariant derivatives in Riemannian geometry, take a form that depends on the representation of the object they act upon.

However, for $\square = D_\tau D^\tau$ we may make use of the very recent finding after (6.5) that $\partial_\nu G^\nu = 0$ for a massive gauge boson *even in non-abelian gauge theory*, and specifically, may add this “zero” to (5.17) and thus write:

$$\begin{aligned} D_\tau D^\tau &= \partial_\tau\partial^\tau - i\partial_\tau G^\tau - iG_\tau\partial^\tau - G_\tau G^\tau = \partial_\tau\left(\partial^\tau - iG^\tau\right) - iG_\tau\left(\partial^\tau - iG^\tau\right) = \left(\partial_\tau - iG_\tau\right)\left(\partial^\tau - iG^\tau\right) \\ &= \partial_\tau\partial^\tau + V \end{aligned}, \quad (6.6)$$

where in the final line we have defined the gauge field *perturbation* (see, e.g., [14] eq. [4.4]):

$$V \equiv -i\left(\partial_\tau G^\tau + G_\tau \partial^\tau\right) - G_\tau G^\tau \overset{i\partial \to k}{\Rightarrow} -k_\tau G^\tau - G_\tau k^\tau - G_\tau G^\tau . \tag{6.7}$$

This use of $\partial_\nu G^\nu = 0$ *does* allow a clean separation $D_\tau D^\tau = \left(\partial_\tau - iG_\tau\right)\left(\partial^\tau - iG^\tau\right)$, and it enables us to explicitly introduce and identify gauge field perturbations. This will be very useful throughout the subsequent development. And again, because we are considering a massive gauge boson, $\partial_\nu G^\nu = 0$ is not just an optional gauge condition; it is required for continuity.

With these preliminaries behind us, it is time to calculate the inverse of (5.15) for a massive gauge boson. We start with the inverse $I_{\mu\nu}$ of (6.2), for which we follow Coleman and Zee and add the Proca mass as follows:

$$I_{\mu\nu}\left(g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu\right) = -\delta^\sigma{}_\mu . \tag{6.8}$$

It is well-known how to calculate inverses of the form (6.8), but we do need to be cognizant of two important points because the D are not the same as ordinary ∂ especially in flat spacetime. First, while $\left[\partial^\sigma, \partial^\nu\right] = 0$ in flat spacetime, we cannot treat $D^\sigma D^\nu$ as commuting here, that is, $\left[D^\sigma, D^\nu\right] \neq 0$. In fact, as noted prior to (6.6), $i\left[D^\sigma, D^\nu\right] = G^{[\sigma}\partial^{\nu]} - 3i\left[G^\sigma, G^\nu\right] \neq 0$ when the operand is G_σ. So we need to be very careful throughout to maintain strict commutation ordering. Second, we cannot just put expressions involving $D^\sigma D^\nu$ or $D_\tau D^\tau$ into a denominator. Rather, we have to treat carefully, as inverses and not mere denominators, inverse expressions which contain $D^\sigma D^\nu$ as well as the gauge-covariant d'Alembertian $\Box = D_\tau D^\tau$.

With that in mind, let us calculate $I_{\mu\nu}$. First, we specify $I_{\mu\nu}$ using the general form with *A* and *B* unknown and to-be-deduced:

$$I_{\mu\nu} \equiv A g_{\mu\nu} + B D_\mu D_\nu . \tag{6.9}$$

Given that $I_{\mu\nu} \neq I_{\nu\mu}$ (to see this, simply note that $D_\mu D_\nu \neq D_\nu D_\mu$), the above definition together with $G_\mu \equiv I_{\mu\nu} J^\nu$ leads to $G_\mu \equiv \left(A g_{\mu\nu} + B D_\mu D_\nu\right) J^\nu = A g_{\mu\nu} J^\nu + B D_\mu D_\nu J^\nu = A J_\mu$ once the continuity relation $D_\nu J^\nu = 0$ of (5.20) is applied. So the inner-product definition $G_\mu \equiv I_{\mu\nu} J^\nu$ combined with the inverse definition (6.9) will eventually allow the important simplification of setting $B D_\mu D_\nu \to 0$ by continuity, which is analogous to what happens in abelian gauge theory when the continuity equation $\partial_\nu J^\nu = 0$ is applied.

So, the task now is to find the unknowns *A* and *B*. If we place (6.9) into (6.8) we obtain:

$$\begin{aligned}
-\delta^{\sigma}{}_{\mu} &= \left(Ag_{\mu\nu}+BD_{\mu}D_{\nu}\right)\left(g^{\nu\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-D^{\sigma}D^{\nu}\right) \\
&= Ag_{\mu\nu}g^{\nu\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-Ag_{\mu\nu}D^{\sigma}D^{\nu}+BD_{\mu}D_{\nu}g^{\nu\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-BD_{\mu}D_{\nu}D^{\sigma}D^{\nu} \; . \\
&= A\delta^{\sigma}{}_{\mu}\left(D_{\tau}D^{\tau}+m^{2}\right)-AD^{\sigma}D_{\mu}+BD_{\mu}D^{\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-BD_{\mu}D_{\nu}D^{\sigma}D^{\nu}
\end{aligned} \tag{6.10}$$

Matching up the terms with $\delta^{\sigma}{}_{\mu}$ we first obtain $-1=A\left(D_{\tau}D^{\tau}+m^{2}\right)$, or inverting:

$$A=-\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}. \tag{6.11}$$

We then use (6.11) in (6.10) and reduce, to next obtain:

$$0=\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}D^{\sigma}D_{\mu}+B\left(D_{\mu}D^{\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-D_{\mu}D_{\nu}D^{\sigma}D^{\nu}\right), \tag{6.12}$$

or, rearranged:

$$B=-\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}D^{\sigma}D^{\alpha}\left(D^{\alpha}D^{\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-D^{\alpha}D_{\tau}D^{\sigma}D^{\tau}\right)^{-1}. \tag{6.13}$$

Finally, we use (6.11) and (6.13) in (6.9) to find that:

$$I_{\mu\nu}=-\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}\left[g_{\mu\nu}+D^{\sigma}D^{\alpha}\left(D^{\alpha}D^{\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-D^{\alpha}D_{\tau}D^{\sigma}D^{\tau}\right)^{-1}D_{\mu}D_{\nu}\right]. \tag{6.14}$$

Above, each derivative pair is defined by $D^{\sigma}D^{\nu}\equiv\partial^{\sigma}\partial^{\nu}-iG^{\sigma}\partial^{\nu}-2G^{\sigma}G^{\nu}+G^{\nu}G^{\sigma}$ in (5.16) and $\Box=D_{\tau}D^{\tau}=\partial_{\tau}\partial^{\tau}-iG_{\tau}\partial^{\tau}-G_{\tau}G^{\tau}$ in (5.17) (remember too, that $\partial_{\tau}G^{\tau}=0$ which produces (6.6) and (6.7)). We may then substitute (6.14) into the original definition $G_{\mu}\equiv I_{\mu\nu}J^{\nu}$ to conclude that:

$$\begin{aligned}
G_{\mu} &= I_{\mu\nu}J^{\nu}=-\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}\left[g_{\mu\nu}+D^{\sigma}D^{\alpha}\left(D^{\alpha}D^{\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-D^{\alpha}D_{\tau}D^{\sigma}D^{\tau}\right)^{-1}D_{\mu}D_{\nu}\right]J^{\nu} \\
&= -\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}g_{\mu\nu}J^{\nu}-\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}D^{\sigma}D^{\alpha}\left(D^{\alpha}D^{\sigma}\left(D_{\tau}D^{\tau}+m^{2}\right)-D^{\alpha}D_{\tau}D^{\sigma}D^{\tau}\right)^{-1}D_{\mu}D_{\nu}J^{\nu} \; . \\
&= -\left(D_{\tau}D^{\tau}+m^{2}\right)^{-1}J_{\mu}=-\left(\partial_{\tau}\partial^{\tau}-iG_{\tau}\partial^{\tau}-G_{\tau}G^{\tau}+m^{2}\right)^{-1}J_{\mu}
\end{aligned} \tag{6.15}$$

In an essential step, we get to the final line by enforcing continuity $D_{\nu}J^{\nu}=0$ from (5.20), and then making use of the d'Alembertian $\Box=D_{\tau}D^{\tau}$ of (5.17). We shall shortly add a term $-i\partial_{\tau}G^{\tau}=0$ to the expression for which the inverse is being taken, so that we can take advantage of (6.6) and explicitly identify the perturbations *V*.

To make all of this appear a bit more familiar to the way such inverses are usually written, let us set $D\to\partial$ in (6.14), and let us assume flat spacetime so all derivatives commute,

$\left[\partial_\mu, \partial_\nu\right]=0$. With these assumptions, the inverses can be treated as regular denominators. With all this, we find embedded in (6.15), the very familiar abelian (*A* subscript) inverse $I_{\mu\nu} \rightarrow I_{A\mu\nu}$:

$$I_{A\mu\nu}=-\frac{g_{\mu\nu}+\dfrac{\partial^\sigma\partial^\alpha\partial_\mu\partial_\nu}{\partial^\alpha\partial^\sigma\left(\partial_\tau\partial^\tau+m^2\right)-\partial^\alpha\partial_\tau\partial^\sigma\partial^\tau}}{\partial_\tau\partial^\tau+m^2}=-\frac{g_{\mu\nu}+\dfrac{\partial_\mu\partial_\nu}{m^2}}{\partial_\tau\partial^\tau+m^2}\overset{i\partial\rightarrow k}{\Rightarrow}\frac{g_{\mu\nu}-\dfrac{k_\mu k_\nu}{m^2}}{k_\tau k^\tau-m^2}\overset{+i\varepsilon}{\Rightarrow}\frac{g_{\mu\nu}-\dfrac{k_\mu k_\nu}{m^2}}{k_\tau k^\tau-m^2+i\varepsilon}. \quad (6.16)$$

With the first arrow, we convert to momentum space via $i\partial_\mu \rightarrow k_\mu$. With the second arrow, we then add the $+i\varepsilon$ prescription. Using the final term above with $G_{A\mu}=I_{A\mu\nu}J^\nu$, we may write:

$$G_{A\mu}=I_{A\mu\nu}J^\nu=\frac{g_{\mu\nu}-\dfrac{k_\mu k_\nu}{m^2}}{k_\tau k^\tau-m^2+i\varepsilon}J^\nu\overset{k_\nu J^\nu=0}{=}\frac{g_{\mu\nu}}{k_\tau k^\tau-m^2+i\varepsilon}J^\nu=\frac{1}{k_\tau k^\tau-m^2+i\varepsilon}J_\mu, \quad (6.17)$$

where $k_\nu J^\nu=i\partial_\nu J^\nu=0$, which is just another version of the continuity equation, is used for the reduction after the third equal sign. If we set $m=0$ in (6.16) we then obtain the clearly indeterminate result:

$$I_{A\mu\nu}=\frac{g_{\mu\nu}-\dfrac{k_\mu k_\nu}{0}}{k_\tau k^\tau+i\varepsilon}=\frac{g_{\mu\nu}-\infty}{k_\tau k^\tau+i\varepsilon}=-\infty. \quad (6.18)$$

But in contrast, doing the same in (6.17) simply yields the finite:

$$G_{A\mu}=\frac{1}{k_\tau k^\tau+i\varepsilon}J_\mu. \quad (6.19)$$

The infinite result in (6.18) is tamed in (6.19) *because of the continuity imposed in (6.17).* If we then put the boson on mass shell, $k_\tau k^\tau=0$, we finally have:

$$G_{A\mu}=\frac{1}{i\varepsilon}J_\mu. \quad (6.20)$$

This only stays finite because of the $+i\varepsilon$ prescription. Equation (6.18) explicitly illustrates why $g^{\nu\sigma}\partial_\tau\partial^\tau-\partial^\sigma\partial^\nu$ has no inverse, or more precisely, why the abelian inverse for a massless gauge boson in flat spacetime is indeterminately-infinite. Equation (6.20) explicitly illustrates why this inverse is also indeterminately-infinite for on-shell bosons, unless one uses the $+i\varepsilon$ prescription.

Now let us do the same in the *non-abelian* inverse (6.14) to see whether the same infinities are encountered. Setting $m=0$ in (6.14) we simply obtain:

$$I_{\mu\nu} = -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + D^\sigma D^\alpha \left(D^\alpha D^\sigma D^\tau D_\tau - D^\alpha D^\tau D^\sigma D_\tau\right)^{-1} D_\mu D_\nu\right]. \tag{6.21}$$

The term $D^\alpha D^\sigma D^\tau D_\tau - D^\alpha D^\tau D^\sigma D_\tau = D^\alpha\left[D^\sigma, D^\tau\right]D_\tau$ must be evaluated using the $D^\sigma D^\nu$ and $D_\tau D^\tau$ of (5.16) and (5.17), that is, as a second order quadratic rather than a fourth order linear term. That is because these derivatives were obtained prior to inversion by operating on G_σ and because the explicit form of a gauge-covariant derivative depends upon its operand. Thus, from (5.16) and (5.17):

$$\begin{aligned} &D^\alpha D^\sigma D^\tau D_\tau - D^\alpha D^\tau D^\sigma D_\tau = \left(D^\alpha D^\sigma\right)\left(D^\tau D_\tau\right) - \left(D^\alpha D^\tau\right)\left(D^\sigma D_\tau\right) \\ &= \left(\partial^\alpha\partial^\sigma - iG^\alpha\partial^\sigma - 2G^\alpha G^\sigma + G^\sigma G^\alpha\right)\left(\partial^\tau\partial_\tau - iG^\tau\partial_\tau - G^\tau G_\tau\right) \\ &-\left(\partial^\alpha\partial^\tau - iG^\alpha\partial^\tau - 2G^\alpha G^\tau + G^\tau G^\alpha\right)\left(\partial^\sigma\partial_\tau - iG^\sigma\partial_\tau - 2G^\sigma G_\tau + G_\tau G^\sigma\right) \end{aligned}. \tag{6.22}$$

If it was possible to commute $\left[D^\sigma, D^\tau\right]=0$, then this term would become zero and (6.21) would contain $\left(D^\alpha\left[D^\sigma, D^\tau\right]D_\tau\right)^{-1} = 0^{-1} = \infty$ and become indeterminate when the mass is zero for the same reason as (6.18). But the defining feature of non-abelian gauge theory is that the gauge fields do not commute, i.e., that $\left[G^\sigma, G^\tau\right]=0$. So the term (6.22) is *not* zero and thus (6.21) does not become infinite *even when the mass is set to zero*. It is the non-commuting nature of non-Abelian gauge theory that bears direct responsibility for maintaining a finite inverse (6.21) for the configuration space operator $g^{\nu\sigma}D_\tau D^\tau - D^\sigma D^\nu$ in (6.1) *even when the gauge boson has no mass*. As we see in (6.15), however, none of this matters at all once we apply $D_\nu J^\nu = 0$ continuity, because that zeroes out the term in (6.22) entirely. Indeed, setting $m=0$ in the non-abelian relation (6.15) for $G(J)$ simply yields

$$G_\mu = -\left(D_\tau D^\tau\right)^{-1} J_\mu = -\left(\partial_\tau\partial^\tau - iG_\tau\partial^\tau - G_\tau G^\tau\right)^{-1} J_\mu . \tag{6.23}$$

Now let us examine what happens for on-shell bosons in non-abelian gauge theory. The relativistic energy relationship is $p_\sigma p^\sigma - m^2 = 0$. Via $\eta^{\sigma\tau} = \frac{1}{2}\left(\gamma^\sigma\gamma^\tau + \gamma^\tau\gamma^\sigma\right) = \frac{1}{2}\left\{\gamma^\sigma, \gamma^\tau\right\}$ this becomes $(\not{p} - m)u = 0 \Leftrightarrow (i\not{\partial} - m)\psi = 0$ when operating on a free, non-interacting Dirac spinor / wavefunction. But for interaction via a gauge field G^τ, $p_\sigma p^\sigma - m^2 = 0$ becomes $\pi_\sigma\pi^\sigma - m^2 = 0$ with $\pi^\tau \equiv p^\tau + G^\tau$ defining the *kinetic momentum* π^τ in relation to the canonical momentum p^τ and the gauge field G^τ. This means that $(\not{\pi} - m)u = (\not{p} + \not{G} - m)u = 0$, or, with $p \to i\partial$ and $u \to \psi$, $(i\not{\partial} + \not{G} - m)\psi = 0$ is Dirac's equation for an *interacting* fermion. The key point of all this – with p^τ and k^σ respectively used to denote fermion and boson momentum vectors – is that a free on-shell fermion is described by $p_\sigma p^\sigma - m^2 = 0$ and a free on-shell gauge boson by

$k_\sigma k^\sigma - m^2 = 0$. But for an interacting on-shell particle with $\pi^\tau \equiv p^\tau + G^\tau$ for fermions and $\pi^\tau \equiv k^\tau + G^\tau$ for bosons, the exact form of the on-shell equation depends on whether G^τ is an abelian or a non-abelian gauge field. Let us see why:

Suppose that G^τ is a U(1) photon / electromagnetic potential A^τ. Here the on-shell relationship, referring also to the perturbation (6.7) and noting that $\pi_\sigma \pi^\sigma = k_\sigma k^\sigma - V$ because $k_\tau G^\tau = 0$, is:

$$\begin{aligned} 0 &= \pi_\sigma \pi^\sigma - m^2 = \left(k_\sigma + A_\sigma\right)\left(k^\sigma + A^\sigma\right) - m^2 = k_\sigma k^\sigma + k_\sigma A^\sigma + A_\sigma k^\sigma + A_\sigma A^\sigma - m^2 \\ &= -V + k_\sigma k^\sigma - m^2 \end{aligned} . \tag{6.24}$$

This perturbation $-V = k_\tau A^\tau + A_\tau k^\tau + A_\tau A^\tau$ is a 1x1 scalar number which can be added to the number $k_\sigma k^\sigma - m^2$, so that (6.24) is a sensible equation. But suppose now that $G^\tau = \lambda^i G^{i\tau}$ is an NxN object formed using the generators λ^i of the simple gauge group SU(N). To be explicit, showing Yang-Mills indexes $A, B = 1 \ldots N$ for the fundamental SU(N) representation, suppose now that $G^\tau{}_{AB} = \lambda^i{}_{AB} G^{i\tau}$. Then, if carelessly generalized, (6.24) would become:

$$\begin{aligned} 0 &= \pi_\sigma \pi^\sigma - m^2 = \left(k_\sigma + G_\sigma\right)\left(k^\sigma + G^\sigma\right) - m^2 = k_\sigma k^\sigma + k_\sigma G^\sigma{}_{AB} + G_{\sigma AB} k^\sigma + \left(G_\sigma G^\sigma\right)_{AB} - m^2 \\ &= -V_{AB} - \delta_{AB}\left(m^2 - k_\sigma k^\sigma\right) \quad \left(= -V + k_\sigma k^\sigma - m^2\right) \end{aligned} . \tag{6.25}$$

But this expression is not quite right. The $k_\sigma k^\sigma - m^2$ is still a scalar number, and because V_{AB} is now taken to be an NxN object for SU(N), the $k_\sigma k^\sigma - m^2$ will occupy the *diagonal* positions in the overall expression (6.25), hence the explicit showing of $\delta_{AB}\left(m^2 - k_\sigma k^\sigma\right)$. At the same time, $-V_{AB} = k_\sigma G^\sigma{}_{AB} + G_{\sigma AB} k^\sigma + \left(G_\sigma G^\sigma\right)_{AB}$ will now be an NxN Hermitian matrix with off-diagonal elements. The perturbation V_{AB} is a matrix, while $k_\sigma k^\sigma - m^2$ is a scalar number that we also know is part of an inverse abelian propagator. So the only way to make sense out of (6.25) is to use this as an *eigenvalue equation* in which $m^2 - k_\sigma k^\sigma$ represents the scalar eigenvalues of the perturbation $-V_{AB}$.

Now, one way to write (6.25) as an eigenvalue equation, it to have it operate on an N-component column vector φ, and to rewrite the non-abelian on-shell condition as $\left[V_{AB} - \delta_{AB}\left(m^2 - k_\sigma k^\sigma\right)\right]\varphi = 0$. But because expressions such as (6.25) will show up in the context of equations such as (6.15), we want to be able to express the on-shell condition independently of any φ. We can do so by taking the determinant $\left|A\right| = \det A$ of (6.25), in the form of the eigenvalue equation:

$$0=\left|\pi_\sigma\pi^\sigma-m^2\right|=-\left|V_{AB}-\delta_{AB}\left(k_\sigma k^\sigma-m^2\right)\right| \quad \left(=\left|-V+k_\sigma k^\sigma-m^2\right|\right). \tag{6.26}$$

This is what specifies an on-shell gauge boson in non-abelian gauge theory. On shell, the eigenvalue solutions of the perturbation V_{AB} are given by the scalar number $k_\sigma k^\sigma - m^2$.

In view of this, if we therefore write (6.15) with $+i\varepsilon$ and $\pi^\tau \equiv k^\tau + G^\tau$ as

$$\begin{aligned} G_\mu &= -\left(D_\tau D^\tau + m^2 - i\varepsilon\right)^{-1} J_\mu = -\left(\partial_\tau\partial^\tau - iG_\tau\partial^\tau - G_\tau G^\tau + m^2 - i\varepsilon\right)^{-1} J_\mu \\ &\overset{i\partial\to k}{\Rightarrow} \left(k_\tau k^\tau + G_\tau k^\tau + G_\tau G^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu \\ &\overset{k_\tau G^\tau=0}{\Rightarrow} \left(k_\tau k^\tau + k_\tau G^\tau + G_\tau k^\tau + G_\tau G^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu = \left(\pi_\tau\pi^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu \\ &= \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu \end{aligned}, \tag{6.27}$$

we see by writing (6.17) in the form of an inverse:

$$G_{A\mu} = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu, \tag{6.28}$$

that the *sole* difference between the abelian and non-abelian solutions for $G_\mu\left(J_\mu\right)$ is that the canonical scalar $k_\tau k^\tau$ of abelian gauge theory is replaced by the kinetic scalar $\pi_\tau\pi^\tau$ in non-abelian gauge theory, or, alternatively and equivalently, that a perturbation $-V=-V_{AB}$ is added to the abelian (6.28) to arrive at the non-abelian (6.27), which then turns the usual inverse propagator $k_\tau k^\tau - m^2 + i\varepsilon$ into $-V_{AB} + k_\tau k^\tau - m^2 + i\varepsilon$ for which on-shell particles are described by $\left|V_{AB}-\delta_{AB}\left(k_\sigma k^\sigma - m^2\right)\right|=0$ in (6.26).

If the "careless" $\pi_\sigma\pi^\sigma - m^2 = 0$ in (6.25) were to describe the on-shell condition for an interacting particle in non-abelian gauge theory – which it does *not* – then for an on-shell particle, (6.27) in the form $G_\mu = \left(\pi_\tau\pi^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ would reduce to $G_\mu = \left(+i\varepsilon\right)^{-1} J_\mu$ which is exactly the same as the abelian (6.20). So in either abelian or non-abelian gauge theory, we would require the $+i\varepsilon$ prescription to avoid the poles for an on-shell particle. However, $\pi_\sigma\pi^\sigma - m^2 = 0$ is *not* the on-shell condition for non-abelian gauge theory. Rather, on-shell bosons are specified by the eigenvalue equation $\left|\pi_\sigma\pi^\sigma - m^2\right| = 0$ of (6.26). So even with $\left|\pi_\sigma\pi^\sigma - m^2\right| = 0$, the expression $G_\mu = \left(\pi_\tau\pi^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ will generally remain finite in non-abelian gauge theory even if we use $G_\mu = \left(\pi_\tau\pi^\tau - m^2\right)^{-1} J_\mu$ absent $+i\varepsilon$. Because on shell particles are described by $\left|\pi_\sigma\pi^\sigma - m^2\right| = 0$ and not $\pi_\sigma\pi^\sigma - m^2 = 0$ in non-abelian gauge theory, the non-abelian theory remains finite on shell even absent $+i\varepsilon$.

Before studying massless gauge bosons using the more formal approach of Faddeev-Popov, we also note that the continuity relation $\partial_\nu J^\nu = 0$ which tames $G_\mu\left(J_\mu\right)$ in the massless boson inverse (6.19) notwithstanding the infinite inverse (6.18), plays a similar role in taming the quantum field amplitude obtained from the QED path integral. Specifically, the action corresponding to the field equation $-J^\nu = \left(g^{\nu\sigma}\left(\partial_\tau\partial^\tau + m^2\right) - \partial^\sigma\partial^\nu\right)G_\sigma$ which is the abelian version of (6.3), for which the inverse was found in (6.16), is:

$$S\left(G\right) = \int d^4x\mathscr{L} = \int d^4x\left[\tfrac{1}{2}G_\nu\left(g^{\nu\sigma}\left(\partial_\tau\partial^\tau + m^2\right) - \partial^\nu\partial^\sigma\right)G_\sigma - G_\sigma J^\sigma\right]. \tag{6.29}$$

When the Gaussian integral $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right) = \left(-2\pi / A\right)^{.5}\exp\left(J^2 / 2A\right)$ is employed as the template to use (6.29) in $Z = \int DG\exp\left(iS\left(G\right)\right) \equiv C\exp\left(iW\left(J\right)\right)$, the inverse in $J^2/2A$ is based on the abelian inverse $I_{A\mu\nu}$ in (6.16), and we obtain (see, e.g., [11], pages 30-31):

$$W\left(J\right) = -\frac{1}{2}\int\frac{d^4k}{\left(2\pi\right)^4}J^\mu\left(k\right)*\frac{-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{m}}{k_\tau k^\tau - m + i\varepsilon}J^\nu\left(k\right) = \frac{1}{2}\int\frac{d^4k}{\left(2\pi\right)^4}J^\mu\left(k\right)*I_{A\mu\nu}J^\nu\left(k\right) . \tag{6.30}$$

This too looks like it will become singular for $m=0$, just like (6.18). But there too, as in (6.17), the continuity relationship $k_\nu J^\nu = i\partial_\nu J^\nu = 0$ rescues the path integral from an indeterminate fate, and facilitates the reduction:

$$W\left(J\right) = +\frac{1}{2}\int\frac{d^4k}{\left(2\pi\right)^4}J^\mu\left(k\right)*\frac{1}{k_\tau k^\tau - m + i\varepsilon}J_\mu\left(k\right) \overset{m=0}{\Rightarrow} +\frac{1}{2}\int\frac{d^4k}{\left(2\pi\right)^4}J^\mu\left(k\right)*\frac{1}{k_\tau k^\tau + i\varepsilon}J_\mu\left(k\right). \tag{6.31}$$

This also tells us that the electromagnetic force between like-charges is repulsive.

But the key feature of interest in both (6.17) which is for a classical field and (6.31) which is for a quantum field, is that even though the *mathematical* abelian inverse (6.16) becomes infinite if $m=0$, when this inverse is placed into the context of a *physical* equation such as $G_{A\mu} = I_{A\mu\nu}J^\nu$ in (6.17) or $...J^\mu * I_{A\mu\nu}J^\nu$ in (6.30), the seemingly-infinite result becomes finite and well-behaved. This is because the physical context – in this case the continuity relation $k_\nu J^\nu = i\partial_\nu J^\nu = 0$ – causes the otherwise singular term $k_\mu k_\nu / m \to k_\mu k_\nu / 0 = \infty$ to be zeroed out *before it ever gets to wreak any havoc*. This contextual finiteness is very important, because even though the mathematical object – the inverse – becomes singular, the physical result remains finite. In the discussion to now be developed, where we use the more formal approach of Faddeev-Popov to develop the massive gauge bosons, this will lead to what we shall call "contextual gauge fixing." In Faddeev-Popov, where a gauge number ξ enables an unlimited array of non-unique inverses, the continuity relation forces the physical results into a very definite and unique choice of gauge. When we use these same inverses in $\oiint F\left(G\left(J\right)\right)$ to show why $\oiint F = -i\iiint dGG$ looks very much like a baryon, this type of "contextual gauge fixing" coupled with Fermi-Dirac-Pauli Exclusion will not only result in unique solutions for $G\left(J\right)$, but will give mass to the fermions of $J^{\mu} = \overline{\psi}\gamma^{\mu}\psi$ and turn them into quarks, while rendering the massive gauge bosons massless just like gluons.

7. Abelian and non-Abelian Massless Gauge Boson Inverses for the Electric Charge Density, Using the Faddeev-Popov Method

In the last section we took the "pragmatic" Coleman-Zee approach of obtaining the classical field equation inverse for a massive gauge boson and then setting the mass to zero to see what happens under a variety of circumstances. Now, we take the more formal, direct approach of using the Faddeev-Popov method to calculate the inverse for a massless gauge boson *ab initio*, without the intermediate stop for a massive boson.

If we take the "non-pragmatic" route and start out with a *massless* gauge boson for which we apply Faddeev-Popov, and to open simplified discussion revert (5.15) to its abelian limit $D \rightarrow \partial$, then along the way the *effective* field equation becomes (see [11], after (III.4(8))):

$$-J^{\nu} = \left(g^{\nu\sigma}\partial_{\tau}\partial^{\tau} - \left(1 - 1/\xi\right)\partial^{\sigma}\partial^{\nu}\right)G_{\sigma}, \tag{7.1}$$

where ξ is a gauge number. While for the moment we treat the introduction of ξ simply as a mathematical manipulation of the classical field equation $-J^{\nu} = \left(g^{\nu\sigma}\partial_{\tau}\partial^{\tau} - \partial^{\sigma}\partial^{\nu}\right)G_{\sigma}$ of (5.15) to which (7.1) reduces for $\xi = \infty$, we keep in mind that ξ actually arises when we start with a path integral $Z = \int DG \exp\left(iS\left(G\right)\right)$ and turn this into $Z = \int DG \exp\left(i\left[S\left(G\right) - \left(i/2\xi\right)\int d^4x\left(\partial G\right)^2\right]\right)$ through a change of the integration variable which maintains the invariance of the Z under the abelian gauge transformation $G \rightarrow G' = G + d\theta$. So by introducing ξ in this way, and knowing that this carries over to non-abelian gauge theory but for the further introduction of ghost fields $c^{\dagger}, c$ with a path integral $Z = \int DGDcDc^{\dagger} \exp\left(i\left[S\left(G\right) - \left(1/2\xi\right)\int d^4x\left(\partial G\right)^2\right] + S\left(c^{\dagger}, c\right)\right)$ containing a ghost action $S\left(c^{\dagger}, c\right)$, we have a "hook" by which this can eventually be used to set up a quantum path integration for non-abelian theory. But for now, as discussed at length in section 4, we continue to develop the classical theory.

Once again using an inner-product definition $G_{A\mu} \equiv I_{A\mu\nu}J^{\nu}$ for the abelian inverse, in flat spacetime we may multiply through by $-I_{A\mu\nu}$ and write (7.1) as (contrast (6.1)):

$$I_{A\mu\nu}J^{\nu} = -I_{A\mu\nu}\left(g^{\nu\sigma}\partial_{\tau}\partial^{\tau} - \left(1 - 1/\xi\right)\partial^{\sigma}\partial^{\nu}\right)G_{\sigma} = G_{\mu} = \delta^{\sigma}{}_{\mu}G_{\sigma} \tag{7.2}$$

from which we extract (contrast (6.2)):

$$I_{A\mu\nu}\left(g^{\nu\sigma}\partial_\tau\partial^\tau-\left(1-1/\xi\right)\partial^\sigma\partial^\nu\right)=-\delta^\sigma{}_\mu\,. \tag{7.3}$$

Then using $I_{A\mu\nu}\equiv Ag_{\mu\nu}+B\partial_\mu\partial_\nu$ based on (6.9), this becomes (contrast (6.10)):

$$\begin{aligned}-\delta^\sigma{}_\mu&=\left(Ag_{\mu\nu}+B\partial_\mu\partial_\nu\right)\left(g^{\nu\sigma}\partial_\tau\partial^\tau-\left(1-1/\xi\right)\partial^\sigma\partial^\nu\right)\\&=Ag_{\mu\nu}g^{\nu\sigma}\partial_\tau\partial^\tau-Ag_{\mu\nu}\left(1-1/\xi\right)\partial^\sigma\partial^\nu+B\partial_\mu\partial_\nu g^{\nu\sigma}\partial_\tau\partial^\tau-B\partial_\mu\partial_\nu\left(1-1/\xi\right)\partial^\sigma\partial^\nu\,.\\&=A\delta^\sigma{}_\mu\partial_\tau\partial^\tau-A\left(1-1/\xi\right)\partial^\sigma\partial_\mu+B\partial_\mu\partial^\sigma\partial_\tau\partial^\tau-B\left(1-1/\xi\right)\partial_\mu\partial_\nu\partial^\sigma\partial^\nu\end{aligned} \tag{7.4}$$

From this we match up the $\delta^\sigma{}_\mu$ terms to find (contrast (6.11)):

$$A=-1/\partial_\tau\partial^\tau\,. \tag{7.5}$$

so that (cf. (6.12)):

$$0=\frac{\left(1-1/\xi\right)\partial^\sigma\partial_\mu}{\partial_\tau\partial^\tau}+B\left(\partial_\mu\partial^\sigma\partial_\tau\partial^\tau-\left(1-1/\xi\right)\partial_\mu\partial_\tau\partial^\sigma\partial^\tau\right), \tag{7.6}$$

or, commuting and cancelling derivatives freely (cf. (6.13)):

$$B=-\frac{\left(1-1/\xi\right)\dfrac{\partial^\sigma\partial_\mu}{\partial_\mu\partial^\sigma\partial_\tau\partial^\tau-\left(1-1/\xi\right)\partial_\mu\partial_\tau\partial^\sigma\partial^\tau}}{\partial_\tau\partial^\tau}=-\frac{\left(\dfrac{1-1/\xi}{1/\xi}\right)\dfrac{1}{\partial_\alpha\partial^\alpha}}{\partial_\tau\partial^\tau}=\frac{\left(1-\xi\right)\dfrac{1}{\partial_\alpha\partial^\alpha}}{\partial_\tau\partial^\tau}\,. \tag{7.7}$$

Thus, using (7.5) and (7.7) in $I_{A\mu\nu}\equiv Ag_{\mu\nu}+B\partial_\mu\partial_\nu$ we obtain (cf. (6.14) and (6.16)):

$$I_{A\mu\nu}=\frac{-g_{\mu\nu}+\left(1-\xi\right)\dfrac{\partial_\mu\partial_\nu}{\partial_\alpha\partial^\alpha}}{\partial_\tau\partial^\tau}\overset{i\partial\to k}{\Rightarrow}-\frac{-g_{\mu\nu}+\left(1-\xi\right)\dfrac{k_\mu k_\nu}{k_\alpha k^\alpha}}{k_\tau k^\tau}\overset{+i\varepsilon}{\Rightarrow}\frac{g_{\mu\nu}-\left(1-\xi\right)\dfrac{k_\mu k_\nu}{k_\alpha k^\alpha}}{k_\tau k^\tau+i\varepsilon}\,. \tag{7.8}$$

We then use this in $G_{A\mu}\equiv I_{A\mu\nu}J^\nu$ to write:

$$G_{A\mu}\equiv I_{A\mu\nu}J^\nu=\frac{g_{\mu\nu}-\left(1-\xi\right)\dfrac{k_\mu k_\nu}{k_\alpha k^\alpha}}{k_\tau k^\tau+i\varepsilon}J^\nu\,. \tag{7.9}$$

Now let us follow two different routes to reduce (7.9). First, let us apply the continuity relation $k_\nu J^\nu=i\partial_\nu J^\nu=0$ as we did in (6.17). This causes (7.9) to become:

$$G_{A\mu}=\frac{g_{\mu\nu}-(1-\xi)\dfrac{k_\mu k_\nu}{k_\alpha k^\alpha}}{k_\tau k^\tau+i\varepsilon}J^\nu \overset{k_\nu J^\nu=0}{\Rightarrow} \frac{g_{\mu\nu}-(1-\xi)0}{k_\tau k^\tau+i\varepsilon}J^\nu=\frac{g_{\mu\nu}}{k_\tau k^\tau+i\varepsilon}J^\nu=\frac{1}{k_\tau k^\tau+i\varepsilon}J_\mu . \tag{7.10}$$

Alternatively, let us embark upon the different path of selecting the Feynman gauge $\xi=1$ in (7.9). Now we have:

$$G_{A\mu}=\frac{g_{\mu\nu}-(1-\xi)\dfrac{k_\mu k_\nu}{k_\alpha k^\alpha}}{k_\tau k^\tau+i\varepsilon}J^\nu \overset{\xi=1}{\Rightarrow} \frac{g_{\mu\nu}-0\dfrac{k_\mu k_\nu}{k_\alpha k^\alpha}}{k_\tau k^\tau+i\varepsilon}J^\nu=\frac{g_{\mu\nu}}{k_\tau k^\tau+i\varepsilon}J^\nu=\frac{1}{k_\tau k^\tau+i\varepsilon}J_\mu , \tag{7.11}$$

which is the exact same result as in (7.10). And both of these are exactly the same as the result in (6.19). These are three routes to the exact same result. In (7.10), the expression $(1-\xi)0$ which emerges from requiring continuity via $k_\nu J^\nu=i\partial_\nu J^\nu=0$ has *forced* this term to be zeroed out. Just as in (6.17) (and analogously in the non-abelian (6.15)), there is no choice other than to zero out the term containing the gauge number ξ. But if we were unaware of continuity, we could get to the same *effective* inverse $I_{A\mu\nu}=g_{\mu\nu}/(k_\tau k^\tau+i\varepsilon)$ *in general*, by the different route of selecting the Feynman gauge $\xi=1$. Importantly, this means that after we find the inverse and then use it in $G_{A\mu}\equiv I_{A\mu\nu}J^\nu$, we are forced into an equation for $G_{A\mu}$ which could be independently arrived at by selecting the Feynman gauge $\xi=1$ for the standalone inverse.

The point here is that for a *massless* gauge boson, there is a complete freedom to select *any* gauge number $-\infty\le\xi\le\infty$ for the inverse $I_{A\mu\nu}$, which means that this inverse is *infinitely non-unique* when regarded as a *mathematical* entity. This is because of the redundancy whereby G_μ contains four degrees of freedom despite the associated massless physical field having only two degrees of freedom. Nevertheless, once we use this inverse in a *physical* equation such as $G_{A\mu}\equiv I_{A\mu\nu}J^\nu$ in (7.9) to (7.11), the continuity equation forces us to fix the gauge of the inverse into $\xi=1$, or more precisely, forces a result that can equivalently be achieved by selecting $\xi=1$ for the standalone inverse before it is ever inserted into $G_{A\mu}\equiv I_{A\mu\nu}J^\nu$. This is a specific example of the "*contextual gauge fixing*" mentioned at the end of section 6, wherein a gauge which is completely non-unique and thus an associated inverse which is also non-unique as a mathematical matter, is forced to be unique *when placed into a physical context*, in this case, the context of a conserved current density enforced by continuity. In this way, we may think of the Feynman gauge as the "continuity gauge," because it *uniquely* fixes the inverse in the exact same manner as does the continuity equation $k_\nu J^\nu=i\partial_\nu J^\nu=0$.

With (7.1) to (7.11) as a backdrop, we return to the field equation (5.15) with $D^\sigma D^\nu$ and $D_\tau D^\tau$ defined as in (5.16) and (5.17) when the operand is G_σ, and introduce the gauge number ξ exactly as we did in (7.1). Thus, we write:

$$-J^{\nu}=\left(g^{\nu\sigma}D_{\tau}D^{\tau}-\left(1-1/\xi\right)D^{\sigma}D^{\nu}\right)G_{\sigma}\,. \tag{7.12}$$

As with (7.1), we treat the introduction of ξ simply as a mathematical manipulation of (5.15) to which (7.12) will revert for $\xi=\infty$, which allows us to solve this classical equation (7.12) for G_{σ} as a function of J^{ν}. Since $Z=\int DGDcDc^{\dagger}\exp\left(i\left[S\left(G\right)-\left(1/2\xi\right)\int d^{4}x\left(\partial G\right)^{2}\right]+S\left(c^{\dagger},c\right)\right)$ is the path integral for non-abelian gauge theory, it should be clear that the inverse obtained from (7.12) will be a useful item to have "on the shelf" when it comes time to try to calculate the non-ghost portion of this path integral. But for now, we are still working classically, so our imminent goal is to solve the classical equation (7.12) for G_{σ} as a function of J^{ν}.

As we have done previously, we use $G_{\mu}\equiv I_{\mu\nu}J^{\nu}$ to define $I_{\mu\nu}$, and then multiply each side of (7.12) by $-I_{\mu\nu}$ to write:

$$I_{\mu\nu}J^{\nu}=-I_{\mu\nu}\left(g^{\nu\sigma}D_{\tau}D^{\tau}-\left(1-1/\xi\right)D^{\sigma}D^{\nu}\right)G_{\sigma}=G_{\mu}=\delta^{\sigma}{}_{\mu}G_{\sigma}\,. \tag{7.13}$$

From this we extract:

$$I_{\mu\nu}\left(g^{\nu\sigma}D_{\tau}D^{\tau}-\left(1-1/\xi\right)D^{\sigma}D^{\nu}\right)=-\delta^{\sigma}{}_{\mu}\,. \tag{7.14}$$

Then we combine the above with (6.9) to write (cf. (6.10) and (7.4)):

$$\begin{aligned}-\delta^{\sigma}{}_{\mu}&=\left(Ag_{\mu\nu}+BD_{\mu}D_{\nu}\right)\left(g^{\nu\sigma}D_{\tau}D^{\tau}-\left(1-1/\xi\right)D^{\sigma}D^{\nu}\right)\\&=Ag_{\mu\nu}g^{\nu\sigma}D_{\tau}D^{\tau}-Ag_{\mu\nu}\left(1-1/\xi\right)D^{\sigma}D^{\nu}+BD_{\mu}D_{\nu}g^{\nu\sigma}D_{\tau}D^{\tau}-BD_{\mu}D_{\nu}\left(1-1/\xi\right)D^{\sigma}D^{\nu}\,.\\&=A\delta^{\sigma}{}_{\mu}D_{\tau}D^{\tau}-A\left(1-1/\xi\right)D^{\sigma}D_{\mu}+BD_{\mu}D^{\sigma}D_{\tau}D^{\tau}-BD_{\mu}D_{\nu}\left(1-1/\xi\right)D^{\sigma}D^{\nu}\end{aligned} \tag{7.15}$$

Here, the reductions used twice earlier (cf. (6.11) to (6.13) and (7.5) to (7.7)) yield:

$$A=-\left(D_{\tau}D^{\tau}\right)^{-1}, \tag{7.16}$$

$$0=\left(1-1/\xi\right)\left(D_{\tau}D^{\tau}\right)^{-1}D^{\sigma}D_{\mu}+BD_{\mu}D^{\sigma}D_{\tau}D^{\tau}-B\left(1-1/\xi\right)D_{\mu}D_{\tau}D^{\sigma}D^{\tau}\,, \tag{7.17}$$

$$B=-\left(1-1/\xi\right)\left(D_{\tau}D^{\tau}\right)^{-1}D^{\sigma}D^{\alpha}\left(D^{\alpha}D^{\sigma}D_{\beta}D^{\beta}-\left(1-1/\xi\right)D^{\alpha}D^{\beta}D^{\sigma}D_{\beta}\right)^{-1}, \tag{7.18}$$

thus leading via $I_{\mu\nu}\equiv Ag_{\mu\nu}+BD_{\mu}D_{\nu}$ from (6.9), to:

$$I_{\mu\nu}=-\left(D_{\tau}D^{\tau}\right)^{-1}\left[g_{\mu\nu}+\left(1-1/\xi\right)D^{\alpha}D^{\beta}\left(D^{\beta}D^{\alpha}D^{\sigma}D_{\sigma}-\left(1-1/\xi\right)D^{\beta}D^{\sigma}D^{\alpha}D_{\sigma}\right)^{-1}D_{\mu}D_{\nu}\right]. \tag{7.19}$$

Reducing (7.19) is a bit tricky because of the inverse. But if we momentarily put the latter inverse into a "denominator" and use a $_\vee$ marker to hold the commutation position of the inverse, all just to aid in visualization, we may reduce this to:

$$\begin{aligned}
I_{\mu\nu} &= -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + \frac{(1-1/\xi)D^\alpha D^\beta{}_\vee D_\mu D_\nu}{D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma + (1/\xi)D^\beta D^\sigma D^\alpha D_\sigma}\right] \\
&= -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + \frac{(\xi-1)D^\alpha D^\beta{}_\vee D_\mu D_\nu}{\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma}\right] \\
&= -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + (\xi-1)D^\alpha D^\beta\left(\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma\right)^{-1} D_\mu D_\nu\right]
\end{aligned} \tag{7.20}$$

where in the middle line we multiply each of the "numerator" and "denominator" by ξ, then in the final line revert to the inverse formulation.

In this form, we see that the redundancy of G_μ with four degrees of freedom to describe a massless field that has two degrees of freedom permits an infinite non-uniqueness $-\infty \le \xi \le \infty$ in the choice of the gauge number, just as it does in abelian gauge theory, see after (7.11). But now, as before, let us insert this inverse (7.20) into $G_\mu = I_{\mu\nu}J^\nu$ to obtain:

$$G_\mu = -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + (\xi-1)D^\alpha D^\beta\left(\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma\right)^{-1} D_\mu D_\nu\right]J^\nu . \tag{7.21}$$

As in (7.10) and (7.11) we now take two routes to reduce (7.21). For the first route, we apply the non-abelian continuity relationship $D_\nu J^\nu = 0$ deduced in (5.20) to obtain:

$$\begin{aligned}
G_\mu &= -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + (\xi-1)D^\alpha D^\beta\left(\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma\right)^{-1} D_\mu D_\nu\right]J^\nu \\
&\overset{D_\nu J^\nu = 0}{\Rightarrow} -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu}J^\nu + (\xi-1)D^\alpha D^\beta\left(\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma\right)^{-1} D_\mu (0)\right] \\
&= -\left(D_\tau D^\tau\right)^{-1} J_\mu
\end{aligned} \tag{7.22}$$

For the second route, we simply select the Feynman gauge $\xi = 1$ in (7.21). Now we obtain:

$$\begin{aligned}
G_\mu &= -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + (\xi-1)D^\alpha D^\beta\left(\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma\right)^{-1} D_\mu D_\nu\right]J^\nu \\
&\overset{\xi=1}{\Rightarrow} -\left(D_\tau D^\tau\right)^{-1}\left[g_{\mu\nu} + (0)D^\alpha D^\beta\left(\xi\left(D^\beta D^\alpha D^\sigma D_\sigma - D^\beta D^\sigma D^\alpha D_\sigma\right) + D^\beta D^\sigma D^\alpha D_\sigma\right)^{-1} D_\mu D_\nu\right]J^\nu \\
&= -\left(D_\tau D^\tau\right)^{-1} J_\mu
\end{aligned} \tag{7.23}$$

These two results (7.22) and (7.23) are exactly the same. So just as in the abelian (7.10) and (7.11), the Feynman gauge acts as a continuity gauge, because when used in the standalone inverse of (7.20), it leads us to the exact same result as the non-abelian continuity relationship $D_\nu J^\nu = 0$. Additionally, if we now return to (6.15) in which we have also employed continuity, and follow the Coleman-Zee approach of setting the gauge field mass $m = 0$, we also find just as in (7.22) and (7.23) that:

$$G_\mu = -\left(D_\tau D^\tau\right)^{-1} J_\mu = -\left(\partial_\tau \partial^\tau - iG_\tau \partial^\tau - G_\tau G^\tau\right)^{-1} J_\mu \tag{7.24}$$

which we have already seen in (6.23), with $D_\tau D^\tau = \partial_\tau \partial^\tau - iG_\tau \partial^\tau - G_\tau G^\tau$ as found in (5.17), see also (6.6) and (6.7) which make use of $\partial_\nu G^\nu = 0$ for a massive gauge boson and so are able to also provide a connection to the perturbation *V*.

So we see that in contextual setting of the continuity relationship $D_\nu J^\nu = 0$, the *unique* solution to the massless non-abelian field equation $-J^\nu = \left(g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu\right) G_\sigma$ of (5.15) is *always* going to be $G_\mu = -\left(D_\tau D^\tau\right)^{-1} J_\mu$. Whether we arrive at (6.23) / (7.24) by starting with a *massive* gauge field, obtaining the inverse, applying continuity, and then setting $m = 0$ via Coleman-Zee; whether we start with a *massless* gauge field, use Faddeev-Popov to find the inverse, and then apply continuity; or whether we start with a *massless* gauge field, use Faddeev-Popov to find the inverse, and then *choose* the Feynman/continuity gauge $\xi = 1$; we will always end up with the same *unique* solution (7.22) to (7.24).

The point is that even for non-abelian gauge theory, while the *mathematical* inverse for a massless gauge field gives us the freedom to select *any* gauge number $-\infty \le \xi \le \infty$, the *physical* continuity condition $D_\nu J^\nu = 0$ forces us to put the inverse into the Feynman gauge. This *contextual gauge fixing* removes the arbitrariness of the mathematical inverse, and forces us into the specific gauge $\xi = 1$ the moment we use the inverse in $G_\mu = I_{\mu\nu} J^\nu$ and then apply $D_\nu J^\nu = 0$.

Before concluding this section, let us compare the non-abelian results (7.22) to (7.24) all of which are equivalent to one another, with the abelian results (7.10) and (7.11) both of which are also equivalent to one another. The chief difference at this point is that we have not yet introduced the $+i\varepsilon$ prescription into the non-abelian inverses. Comparing (7.22) to (7.24) with (7.10) and (7.11), we see that the way to introduce $+i\varepsilon$ is to amend (7.24) as such:

$$G_\mu = -\left(D_\tau D^\tau - i\varepsilon\right)^{-1} J_\mu = -\left(\partial_\tau \partial^\tau - iG_\tau \partial^\tau - G_\tau G^\tau - i\varepsilon\right)^{-1} J_\mu \overset{i\partial\to k}{\Rightarrow} \left(k_\tau k^\tau + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\mu . \tag{7.25}$$

Above, we have also gone over into momentum space via $i\partial \to k$. This is just the second line of (6.27) with $m = 0$. In the $k_\tau G^\tau = 0$ gauge, which for a massless boson is a choice and not a requirement, this becomes:

$$G_{\mu}=\left(k_{\tau}k^{\tau}+k_{\tau}G^{\tau}+G_{\tau}k^{\tau}+G_{\tau}G^{\tau}+i\varepsilon\right)^{-1}J_{\mu}=\left(\pi_{\tau}\pi^{\tau}+i\varepsilon\right)^{-1}J_{\mu}=\left(-V+k_{\tau}k^{\tau}+i\varepsilon\right)^{-1}J_{\mu}. \tag{7.26}$$

In contrast, if we write (7.10) / (7.11) in the form of an inverse relation, these become:

$$G_{A\mu}=\left(k_{\tau}k^{\tau}+i\varepsilon\right)^{-1}J_{\mu}, \tag{7.27}$$

which is just (6.28) with $m=0$. Of course, the abelian $\left(k_{\tau}k^{\tau}+i\varepsilon\right)^{-1}$ can be written as an ordinary denominator, while the non-abelian $\left(k_{\tau}k^{\tau}+i\varepsilon+G_{\tau}k^{\tau}+G_{\tau}G^{\tau}\right)^{-1}$ cannot because the $G_{\tau}\pi^{\tau}=G_{\tau}k^{\tau}+G_{\tau}G^{\tau}$ term in general will have a matrix form which must be inverted rather than placed in a denominator.

Insofar as on-shell bosons are concerned, as noted in (6.28) and the discussion following, an on-shell boson in non-abelian gauge theory will be described by the eigenvalue equation (6.26), which for $m=0$ and using (6.7) and $\pi_{\sigma}\pi^{\sigma}=k_{\sigma}k^{\sigma}-V$ in the $k_{\tau}G^{\tau}=0$ gauge becomes:

$$0=\left|\pi_{\sigma}\pi^{\sigma}\right|=-\left|V_{AB}-\delta_{AB}k_{\sigma}k^{\sigma}\right| \quad \left(=\left|-V+k_{\sigma}k^{\sigma}\right|=\left|k_{\sigma}k^{\sigma}+k_{\tau}G^{\tau}+G_{\tau}k^{\tau}+G_{\tau}G^{\tau}\right|\right). \tag{7.28}$$

Note again that while $i\partial_{\tau}G^{\tau}=k_{\tau}G^{\tau}=0$ is a *required* relation for a *massive* gauge boson as found in (6.5) and the ensuing discussion, it is an *optional gauge condition* for a *massless* gauge boson. So the relation $G_{\mu}=\left(k_{\tau}k^{\tau}+G_{\tau}k^{\tau}+G_{\tau}G^{\tau}\right)^{-1}J_{\mu}=\left(k_{\tau}k^{\tau}-V\right)^{-1}J_{\mu}$ without mass, whenever it is used, *assumes* the gauge condition $k_{\tau}G^{\tau}=0$. With this gauge condition this can also be written in terms of the kinetic momentum as $G_{\mu}=\left(\pi_{\tau}\pi^{\tau}\right)^{-1}J_{\mu}=\left(k_{\tau}k^{\tau}-V\right)^{-1}J_{\mu}$ and it will not become singular even on-shell because $\left|\pi_{\sigma}\pi^{\sigma}\right|=0$ above, and not $\pi_{\sigma}\pi^{\sigma}=0$, is the on-shell condition for a massless gauge boson in non-abelian theory in the chosen, not required, $k_{\tau}G^{\tau}=0$ gauge. This does introduce a degree of non-uniqueness into the inverse relationship for a massless gauge boson even with continuity which, unlike the residual gauge condition $D_{\nu}D^{\nu}\theta=0$ a.k.a. $\partial_{\nu}\partial^{\nu}\theta-i\partial_{\nu}\left[G^{\nu},\theta\right]=0$ discussed after (6.5), *does* affect the form of the equations whenever one wishes to write them with the perturbation *V*. As such, we will wish to find ways to avoid situations in which $i\partial_{\tau}G^{\tau}=k_{\tau}G^{\tau}=0$ is an optional gauge condition, in favor of always having it be a required relationship.

8. The Recursive Nature of Non-Abelian Gauge Theory, and what it may Teach about Quantizing Yang-Mills Gauge Theory

Now we look for the first time at a very important *recursive* feature of non-abelian gauge theory. If we write the massive boson solution as $G_{\mu}=\left(k_{\tau}k^{\tau}-m^{2}+i\varepsilon+G_{\tau}k^{\tau}+G_{\tau}G^{\tau}\right)^{-1}J_{\mu}$ from the second line of (6.27) and recognize that the perturbation $V = -k_\tau G^\tau - G_\tau k^\tau - G_\tau G^\tau$ in (6.7) may also be written as $V = -G_\tau k^\tau - G_\tau G^\tau$ because $k_\tau G^\tau = 0$ is a *required* condition for a massive gauge boson, see (6.5) et seq., then a preferred way to write and use (6.27) will be the following:

$$G_\mu = \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\mu = \left(k_\tau k^\tau - m^2 + i\varepsilon - V\right)^{-1} J_\mu . \tag{8.1}$$

Again, it bears emphasis, this uses the fact that $k_\tau G^\tau = 0$ is required, but *only* for a massive, not massless, gauge boson. Now, although (8.1) appears on the surface to solve for $G_\mu\left(J_\mu\right)$, this is not a *closed* solution. Rather, it is really a *recursive* solution for $G_\tau\left(G_\tau, J_\tau\right)$ which can be recursed into itself *ad infinitum*. Let us see exactly how this is done.

To do recursion, one generally needs two inputs: first, a recursive kernel; second, a terminal condition. A quintessential example is the recursive definition of the factorial function: The recursive kernel says that $n! = n \times (n-1)!$. The terminal condition says that $0! = 1$. We shall pursue a similar approach to understand G_μ in (8.1).

To keep track of things, let us develop some notations. We shall generally use the double-nested symbol $((\ \ ))$ to denote a recursion. If we recurse G_μ into itself *n* times, we shall denote this as $G_\mu\left((\ \ )\right)_n$. If after *n* recursions we leave the perturbation *V* in the equation, then we shall write this as $G_\mu\left((V)\right)_n$. If, however, after *n* recursive iterations we set $V = 0$, then we shall write this as $G_\mu\left((0)\right)_n \equiv G_\mu\left((V=0)\right)_n$. So, at the zeroth order of recursion, we simply set $-V = G_\tau k^\tau + G_\tau G^\tau = 0$ in (8.1) which removes all of the terms containing G_τ and reduces (8.1) to

$$G_\mu\left((0)\right)_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu . \tag{8.2}$$

This is simply the abelian solution (6.28).

But now, let us perform the first order of recursion. Here, we substitute (8.1) back into itself one time and then set $V = -G_\tau k^\tau - G_\tau G^\tau = 0$. This exercise yields:

$$G_\mu\left((V)\right)_1 = \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\mu$$
$$= \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\mu , \qquad (8.3)$$
$$\overset{V=0}{\Rightarrow} G_\mu\left((0)\right)_1 = \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\mu$$

In leading order, this solution of course still contains (8.2) which is $\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$. But inside the overall inverse we now also have a new $J_\tau k^\tau$ (J^1) and a new $J_\tau J^\tau$ (J^2) term. This is now an expression strictly for $G_\mu\left(J_\mu\right)$ not $G_\tau\left(G_\tau, J_\tau\right)$, because we have cut off the recursion at the first iteration by setting the perturbation $V = -G_\tau k^\tau - G_\tau G^\tau = 0$ in the final line.

Now, let us go to the second order of recursion. Here, we start with the middle line of (8.3), do a second substitution of (8.1) to arrive at the second order recursion, and then cut things off by setting the perturbation $V = 0$. Now we obtain:

$$G_\mu\left((V)\right)_2 = \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\mu$$
$$= \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon \\ + \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\tau k^\tau \\ + \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\tau \\ \times \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J^\tau \end{pmatrix}^{-1} J_\mu ,(8.4)$$

which, upon setting $V = -G_\tau k^\tau - G_\tau G^\tau = 0$ reduces to:

$$G_\mu\left(\left(0\right)\right)_2 = \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon \\ +\left(k_\tau k^\tau - m^2 + i\varepsilon + \dfrac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \dfrac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right)^{-1} J_\tau k^\tau \\ +\left(k_\tau k^\tau - m^2 + i\varepsilon + \dfrac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \dfrac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right)^{-1} J_\tau \\ \times\left(k_\tau k^\tau - m^2 + i\varepsilon + \dfrac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \dfrac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\mu, \tag{8.5}$$

It will be appreciated this second recursive iteration contain terms in *J*, J^2, J^3 and J^4. A third iteration would be expected to produce terms up to J^6, and in general, *n* iterations should produce terms over the entire gamut of $J^1 ... J^{2n}$. As with (8.3), $G_\mu\left(\left(0\right)\right)_2$ is an expression strictly for $G_\mu\left(J_\mu\right)$ (really, $G_\mu\left(J_\mu, k_\mu\right)$), not $G_\tau\left(G_\tau, J_\tau\right)$ because we have cut off the recursion at the second iteration by setting the perturbation $V = 0$. But, having done two iterations rather than one, we have some new terms that we did not have at the first iteration. So in general the technique is to iterate as many times as one wishes, and then set $V = 0$ to end the recursion. Each iteration will add new terms of yet higher order in *J*, and the result will be an expression for $G_\mu\left(J_\mu\right)$ with terms of order $J^1 ... J^{2n}$. And, of course, mathematically, theoretically, to obtain an *exact*, *closed* expression for $G_\mu\left(J_\mu\right)$ not $G_\tau\left(G_\tau, J_\tau\right)$, one would iterate an *infinite* number of times and then set $V = 0$. But, of course, the real method we now need to pursue is not to iterate to infinity, but to figure out the pattern.

To discern the overall pattern, we do one more recursion to the *n*=3 level by substituting (8.1) into the each and every G_μ in (8.4). The expression for $G_\mu\left(\left(V\right)\right)_3$ takes up over a page, and is not shown here. But upon setting $V = 0$ to arrive at $G_\mu\left(\left(0\right)\right)_3$, this reduces to the still very large expression:

$$G_\mu\left(\left(0\right)\right)_3 = \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon \\ + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau k^\tau \\ + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau \\ \times \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau k^\tau \\ + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon +\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau k^\tau \\ + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau \\ \times \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau \\ \times \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau k^\tau \\ + \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J_\tau \\ \times \left(\begin{array}{l} k_\tau k^\tau - m^2 + i\varepsilon + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\tau \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J^\tau \end{array} \right)^{-1} J^\tau \end{array} \right) J^\tau \end{array} \right)^{-1} J_\mu . \quad (8.6)$$

Even this is rather formidable, but now we have enough information to establish a definite pattern that can be generalized to any order of recursion.

Recognizing that the abelian boson propagator π may be denoted $\pi^{-1} \equiv k_\tau k^\tau - m^2 + i\varepsilon$ up to a factor of *i*, we rewrite the abelian (8.2) simply as:

$$G_\mu\left(\left(0\right)\right)_0 = \pi J_\mu . \quad (8.7)$$

We also use this to write (8.3) as:

$$G_\mu\left((0)\right)_1 = \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\mu , \tag{8.8}$$

and to write (8.5) as:

$$G_\mu\left((0)\right)_2 = \begin{pmatrix} \pi^{-1} + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau k^\tau \\ + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\mu , \tag{8.9}$$

Now we see that $\left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1}$ from (8.8) appears three times in (8.9). Given this, let us next define $\Pi^{-1} \equiv \pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau$. This allows us to rewrite (8.8) as:

$$G_\mu\left((0)\right)_1 = \Pi J_\mu , \tag{8.10}$$

and (8.9) as:

$$G_\mu\left((0)\right)_2 = \left(\pi^{-1} + \Pi J_\tau k^\tau + \Pi J_\tau \Pi J^\tau\right)^{-1} J_\mu , \tag{8.11}$$

Now we see that (8.11) looks just like (8.8), except that each π which is in a term with J has advanced to a Π. So now let's go to that rather large (8.6) to nail down the pattern. Using $\pi^{-1} \equiv k_\tau k^\tau - m^2 + i\varepsilon$ we first reduce (8.6) to:

$$G_\mu\left((0)\right)_3 = \begin{pmatrix} \pi^{-1} + \begin{pmatrix} \pi^{-1} + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau k^\tau \\ + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\tau k^\tau \\ + \begin{pmatrix} \pi^{-1} + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau k^\tau \\ + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J_\tau \\ \times \begin{pmatrix} \pi^{-1} + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau k^\tau \\ + \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J_\tau \left(k\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)^{-1} J^\tau \end{pmatrix}^{-1} J^\tau \end{pmatrix}^{-1} J_\mu . \tag{8.12}$$

Now, using $\Pi^{-1} \equiv \left(\pi^{-1} + \pi J_\tau k^\tau + \pi J_\tau \pi J^\tau\right)$, we may further reduce (8.12) to:

$$G_\mu\left((0)\right)_3=\begin{pmatrix}\pi^{-1}+\left(\pi^{-1}+\Pi J_\tau k^\tau+\Pi J_\tau \Pi J^\tau\right)^{-1}J_\tau k^\tau \\ +\left(\pi^{-1}+\Pi J_\tau k^\tau+\Pi J_\tau \Pi J^\tau\right)^{-1}J_\tau\left(\pi^{-1}+\Pi J_\tau k^\tau+\Pi J_\tau \Pi J^\tau\right)^{-1}J^\tau\end{pmatrix}^{-1}J_\mu . \tag{8.13}$$

But now, we see that $\left(\pi^{-1}+\Pi J_\tau k^\tau+\Pi J_\tau \Pi J^\tau\right)^{-1}$ from (8.11) appears three times in (8.13). So now, we define yet another $\overline{\Pi}^{-1}\equiv\pi^{-1}+\Pi J_\tau k^\tau+\Pi J_\tau \Pi J^\tau$ and use this to rewrite (8.11) as:

$$G_\mu\left((0)\right)_2=\overline{\Pi}J_\mu \tag{8.14}$$

and (8.13) as:

$$G_\mu\left((0)\right)_3=\left(\pi^{-1}+\overline{\Pi}J_\tau k^\tau+\overline{\Pi}J_\tau\overline{\Pi}J^\tau\right)^{-1}J_\mu\equiv\overline{\overline{\Pi}}J_\mu . \tag{8.15}$$

This now has the form of (8.11) but with $\Pi\to\overline{\Pi}$. Seeing the pattern, we further define $\overline{\overline{\Pi}}^{-1}=\pi^{-1}+\overline{\Pi}J_\tau k^\tau+\overline{\Pi}J_\tau\overline{\Pi}J^\tau$. It is clear that this is the pattern which will continue for higher recursive order. Now, let us systematize this pattern.

Pulling together the various results from (8.7), (8.10), (8.14), (8.15) and the various notational definitions made along the way, we have:

$$\begin{aligned}G_\mu\left((0)\right)_0&=\pi J_\mu=\left(k_\tau k^\tau-m^2+i\varepsilon\right)^{-1}J_\mu\\ G_\mu\left((0)\right)_1&=\Pi J_\mu=\left(\pi^{-1}+\pi J_\tau k^\tau+\pi J_\tau\pi J^\tau\right)^{-1}J_\mu\\ G_\mu\left((0)\right)_2&=\overline{\Pi}J_\mu=\left(\pi^{-1}+\Pi J_\tau k^\tau+\Pi J_\tau\Pi J^\tau\right)^{-1}J_\mu\\ G_\mu\left((0)\right)_3&=\overline{\overline{\Pi}}J_\mu=\left(\pi^{-1}+\overline{\Pi}J_\tau k^\tau+\overline{\Pi}J_\tau\overline{\Pi}J^\tau\right)^{-1}J_\mu\end{aligned} . \tag{8.16}$$

Of course, for notational economy we do not want to have to keep adding bars or primes or any other qualifier to each of the “propagators.” So let us denote each “propagator” with a subscript that simply declares its recursive order, thus, $\pi\equiv\pi_0$, $\Pi\equiv\pi_1$, $\overline{\Pi}\equiv\pi_2$, $\overline{\overline{\Pi}}\equiv\pi_3$, etcetera. Then, we can inductively compact (8.16) into a fully recursive solution just like the recursive kernel $n!=n\times(n-1)!$ and the terminal condition $0!=1$ for factorial. Specifically, starting with $G_\mu\left((0)\right)_3$ and working down, the recursive kernel and the terminal condition are induced to be:

$$\begin{cases}G_\mu\left((0)\right)_n=\pi_n J_\mu=\left(\pi_0{}^{-1}+\pi_{n-1}J_\tau k^\tau+\pi_{n-1}J_\tau\pi_{n-1}J^\tau\right)^{-1}J_\mu\\ G_\mu\left((0)\right)_0=\pi_0 J_\mu=\left(k_\tau k^\tau-m^2+i\varepsilon\right)^{-1}J_\mu\end{cases} . \tag{8.17}$$

If we wish to separate the propagators from the gauge fields in (8.17), the recursive kernel and the abelian terminal condition may be written also as:

$$\begin{cases} \pi_n = \left(\pi_0{}^{-1} + \pi_{n-1} J_\tau k^\tau + \pi_{n-1} J_\tau \pi_{n-1} J^\tau\right)^{-1} \\ \pi_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} \end{cases} . \tag{8.18}$$

So with all of this in mind, let us now return to (8.1) which is an expression for $G(G,J)$. But at any recursive order, we now know how to turn this into $G(J)$ without any gauge field residual: Just zero out the perturbation. Of course, nature will not stop at some order and then zero out perturbations. She will recurse ad infinitum and the physics we observe will be for an infinite-order recursion. So in the natural world, we expect that the observed non-linear solution for $G(J)$ will be the one which recurses to infinity, thus contains terms up to infinite order in J and in *k* (really, $2\times\infty$ in *J*), and then sets the perturbation *V* to zero. That is, we expect that nature's *physical* solution (8.1) will be:

$$\begin{aligned} G_\mu &= \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\mu = \left(k_\tau k^\tau - m^2 + i\varepsilon - V\right)^{-1} J_\mu \\ &= -\left(D_\tau D^\tau + m^2 - i\varepsilon\right)^{-1} J_\mu = -\left(\partial_\tau \partial^\tau + m^2 - i\varepsilon - iG_\tau \partial^\tau - G_\tau G^\tau\right)^{-1} J_\mu \qquad , \\ &\equiv G_\mu\left((0)\right)_\infty = \pi_\infty J_\mu = \left(\pi_0{}^{-1} + \pi_{\infty-1} J_\tau k^\tau + \pi_{\infty-1} J_\tau \pi_{\infty-1} J^\tau\right)^{-1} J_\mu \end{aligned} \tag{8.19}$$

Above, for future use in doing an analytical path integral in section 11, we have also included the earlier solution (6.27) to the field equation $-J^\nu = \left(g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu\right) G_\sigma$ of (5.15) with a Proca massive boson and $D^\sigma D^\nu \equiv \partial^\sigma \partial^\nu - iG^\sigma \partial^\nu - 2G^\sigma G^\nu + G^\nu G^\sigma$ from (5.16) and $D_\tau D^\tau = \partial_\tau \partial^\tau - iG_\tau \partial^\tau - G_\tau G^\tau$ from (5.17). We especially wish to take note of the correspondence $\pi_\infty \leftrightarrow -\left(D_\tau D^\tau + m^2 - i\varepsilon\right)^{-1}$. And we also note the embedded correspondences $G_\tau k^\tau \leftrightarrow \pi_{\infty-1} J_\tau k^\tau$ and $G_\tau G^\tau \leftrightarrow \pi_{\infty-1} J_\tau \pi_{\infty-1} J^\tau$, which both contain the elemental correspondence $G_\tau \leftrightarrow \pi_{\infty-1} J_\tau \cong \pi_\infty J_\tau$.

Very importantly, written as:

$$\begin{cases} G_\mu = G_\mu\left((0)\right)_\infty = \pi_\infty J_\mu = \left(\pi_0{}^{-1} + \pi_{\infty-1} J_\tau k^\tau + \pi_{\infty-1} J_\tau \pi_{\infty-1} J^\tau\right)^{-1} J_\mu \\ \pi_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} \end{cases} , \tag{8.20}$$

we have an expression for $G(J,k)$ rather than $G(G,J,k)$, with all gauge fields removed. What is left of the gauge field is its momentum vector *k*, interacting with the current density in the terms $J_\tau k^\tau$ and contracted with itself in the linear terms $k_\tau k^\tau$.

Why is this all so very important? First, it points out that although (8.1) appears on the surface to solve for $G_\mu\left(J_\mu\right)$, this is not a *closed* solution. Rather, it is really a *recursive* solution for $G_\tau\left(G_\tau, J_\tau\right)=G_\tau\left(G_\tau\left(G_\tau, J_\tau\right), J_\tau\right)=G_\tau\left(G_\tau\left(G_\tau\left(G_\tau, J_\tau\right), J_\tau\right), J_\tau\right)$... which can be iteratively recursed *ad infinitum*, but at any order can be cut off and turned into $G_\mu\left(J_\mu\right)$ not $G_\tau\left(G_\tau, J_\tau\right)$ by setting $V=0$, i.e., by ceasing any further perturbations. This makes the non-linear nature of Yang-Mills theory very apparent from a different view than $F^k{}_{\mu\nu}=\partial_{[\mu}G^k{}_{\nu]}+f^{ijk}G^i{}_\mu G^j{}_\nu$ of (1.9) or $F=dG+G\wedge G$ of (1.11) which are the usual expressions used to highlight the non-linear nature of Yang-Mills theory.

Secondly, and of very deep importance, this recursion may well point the way toward being able to *analytically and exactly* quantize Yang-Mills theory. Specifically, we now return to Jaffe and Witten who on page 7 of [6], state:

> "Since the inception of quantum field theory, two central methods have emerged to show the existence of quantum fields on non-compact configuration space (such as Minkowski space). These known methods are (i) Find an exact solution in closed form; (ii) Solve a sequence of approximate problems, and establish convergence of these solutions to the desired limit."

The foregoing suggests a third method which is really a hybrid of (i) and (ii): find an exact *recursive kernel* in closed form (which is $G_\mu=\left(k_\tau k^\tau-m^2+i\varepsilon+G_\tau k^\tau+G_\tau G^\tau\right)^{-1}J_\mu$) and then expand that kernel in successive iterations to see how the recursion behaves in the limit of infinite recursive nesting. That is exactly what we have done in (8.17), (8.18) and (8.20).

Specifically, regarding $G_\mu=\left(k_\tau k^\tau-m^2+i\varepsilon+G_\tau k^\tau+G_\tau G^\tau\right)^{-1}J_\mu$ as the zero$^{\text{th}}$ order solution for $G_\tau\left(G_\tau, J_\tau\right)$, with each iteration of $G_\tau\left(G_\tau, J_\tau\right)$ from the n^{th} to the $(n+1)^{\text{th}}$ recursive order we are effectively replacing all gauge fields G_τ at the n^{th} order with current densities J_τ up to the $2(n+1)^{\text{th}}$ order, and at the same time injecting a new set of gauge fields G_τ at the $(n+1)^{\text{th}}$ order. But at any time we can stop introducing new gauge fields by simply setting the perturbation to zero. So at each order, whenever we decide to do so, we may effectively strip out the gauge fields and replace them with current densities. *This means that in the limit $n\to\infty$ we may effectively replace all gauge fields with current densities by stopping perturbation at $n=\infty$.*

Very similarly, when we take a path integral $Z=\int DG\exp iS\left(G\right)=\mathcal{C}\exp iW\left(J\right)$, because G is the integration variable, we effectively strip off the G and obtain a quantum amplitude $W\left(J\right)$ expressed in terms of the current density J. So the infinite recursion has the same effect as a path integral in terms of trading G for J. But as pointed out at the start of section 6, the mathematical exercise of analytically calculating a path integral revolves around clever extrapolations of the Gaussian integral $\int dx\exp\left(-\tfrac{1}{2}Ax^2-Jx\right)=\left(-2\pi/A\right)^{.5}\exp\left(J^2/2A\right)$

into $Z=\int DG\exp iS(G)=\mathcal{C}\exp iW(J)$. The calculation impediment we run into is that $\int dx\exp\left(-\frac{1}{2}Ax^2-Jx\right)$ is integrable because it is quadratic, but becomes quite intractable once this integral involves a polynomial of x^3 and higher order, which is exactly what happens in Yang-Mills theory and indeed, any non-linear interaction theory. Why is this intractable? Because nobody knows how to calculate such integrals exactly and analytically!

The usual and best workaround is to employ what Zee [11] in Appendix A refers to as the "central identity of quantum field theory":

$$\int D\phi\exp\left(-\tfrac{1}{2}\phi\cdot K\cdot\phi-V(\phi)+J\cdot\phi\right)=\mathcal{C}\exp\left(-V(\delta/\delta J)\right)\exp\left(\tfrac{1}{2}J\cdot K^{-1}\cdot J\right). \tag{8.25}$$

This method uses the functional variation $G_\mu\to\delta/\delta J^\mu$ to remove all terms which are polynomial (greater than second order) in the gauge field G_μ, and replace them with terms $\delta/\delta J^\mu$ that contain only the current density. This allows $\exp\left(V(\delta/\delta J)\right)$ to be removed from inside the integral, so that the only terms left inside the integral are quadratic in G_μ. Then, the integral is performed to obtain $\exp\left(\frac{1}{2}J\cdot K^{-1}\cdot J\right)$, and the operation of $\exp\left(-V(\delta/\delta J)\right)$ on $\exp\left(\frac{1}{2}J\cdot K^{-1}\cdot J\right)$ is thereafter used to extract order-by-order terms in the quantum amplitude to reveal various Green's and Wick's coefficients in this amplitude.

The very important point is that an infinitely-iterative application of the recursive kernel $G_\mu=\left(k_\tau k^\tau-m^2+i\varepsilon+G_\tau k^\tau+G_\tau G^\tau\right)^{-1}J_\mu$ of (8.1) serves a purpose totally analogous to $G_\mu\to\delta/\delta J^\mu$. *But* $G_\mu\to\left(\pi_0^{-1}+\pi_{\infty-1}J_\tau k^\tau+\pi_{\infty-1}J_\tau\pi_{\infty-1}J^\tau\right)^{-1}J_\mu$ *from (8.20) is now the replacement we use in lieu of* $G_\mu\to\delta/\delta J^\mu$. In the limit of infinite recursion, this will allow us in section 11 to do an analytically-exact calculation of the path integral by turning G_μ into J_μ on an order-by-order basis such that in the limit of infinite nesting, all of the gauge fields have been replaced by current densities which then pose no problem to carrying out a Gaussian integration which is simply of quadratic form $\int dx\exp\left(-\frac{1}{2}Ax^2-Jx\right)$ in the gauge fields.

Now, let us return to the Yang-Mills monopoles $\oiint F=-i\iiint dGG\neq 0$ of (3.3) and (5.9), and particularly the identity $P'=d[G,G]=dGG$ of (2.11) upon which this is based. It will be our goal to use one or more of the inverses $G(J)$ that we have developed here to replace each G in this monopole with its source current J, then to replace each J with fermions via $J^\mu=\overline{\psi}\gamma^\mu\psi$, then to apply exclusion to the fermions, and then to show that this faux magnetic charge $P'=d[G,G]=dGG$ – at least in the classical theory – has the exact same chromodynamic symmetries as a baryon.

9. Populating the Composite Yang-Mills Magnetic Monopoles with Chromodynamically-Colored Fermions

Let us start the present discussion with the identity $d[G,G]=dGG$ uncovered in (2.11), which we combine with (3.3) and then expand into tensor component expressions (see also (2.8) and (2.9)) while also including the faux magnetic charge $P'=-idGG=-id[G,G]$, as such:

$$\begin{aligned}
\oiint F &= \iiint P' = -i\iiint dGG = -i\iiint d[G,G] = -i\oiint [G,G] \\
&= \oiint \tfrac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = \iiint \tfrac{1}{3!} P'_{\sigma\mu\nu} dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\iiint \tfrac{1}{3!}\left(\partial_{[\sigma} G_{\mu]} G_\nu + \partial_{[\mu} G_{\nu]} G_\sigma + \partial_{[\nu} G_{\sigma]} G_\mu\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\iiint \tfrac{1}{3!}\left(\partial_\sigma \left[G_\mu, G_\nu\right] + \partial_\mu \left[G_\nu, G_\sigma\right] + \partial_\nu \left[G_\sigma, G_\mu\right]\right) dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\oiint \tfrac{1}{2!}\left[G_\mu, G_\nu\right] dx^\mu \wedge dx^\nu \neq 0
\end{aligned} \quad . \qquad (9.1)$$

Let us now further develop (9.1) using the inverses reviewed in sections 6 and 7.

For a *massless* gauge boson in non-abelian gauge theory, we found that the relationship $G_\mu = -\left(D_\tau D^\tau\right)^{-1} J_\mu$ is the *unique* solution to the field equation $-J^\nu = \left(g^{\nu\sigma} D_\tau D^\tau - D^\sigma D^\nu\right) G_\sigma$ of (5.15) with $D^\sigma D^\nu$ and $D_\tau D^\tau$ given by (5.16) and (5.17), in the circumstance where the current density is conserved according to $D_\nu J^\nu = 0$ as found in (5.20), because this continuity contextually fixes the gauge to the Feynman / continuity gauge $\xi = 1$, see (7.22) and (7.23). We further found in (7.24) that by setting the mass $m = 0$ in (6.15) for a *massive* gauge boson, we arrive at exactly the same solution $G_\mu = -\left(D_\tau D^\tau\right)^{-1} J_\mu$. And, we found that in (7.25), in order to include the $+i\varepsilon$ prescription in the non-Abelian theory, we need simply migrate $D_\tau D^\tau \Rightarrow D_\tau D^\tau - i\varepsilon$. So as shown in (6.27), the non-abelian solution for a massive gauge boson is $G_\mu = \left(\pi_\tau \pi^\tau - m + i\varepsilon\right)^{-1} J_\mu$, while as shown in (6.28), the corresponding abelian solution for a massive gauge boson is $G_{A\mu} = \left(k_\tau k^\tau - m + i\varepsilon\right)^{-1} J_\mu$. So again, we are reminded that the non-abelian solution is identical in form to the abelian relation for a massive gauge boson, but for the replacement of the canonical $k_\tau k^\tau$ with the kinetic $\pi_\tau \pi^\tau$ momentum scalar, which replacement can be made in the massive theory because $\partial_\tau G^\tau = 0$ is a requirement, and which replacement *may* be made in the massless theory if one chooses $\partial_\tau G^\tau = 0$ although one does not have to. So the massive solution is more unique in this way than the massless solution.

Now we wish to replace each G_μ in (9.1) with its unique continuity solution, i.e., with the gauge contextually fixed to $\xi = 1$ because of requiring continuity, either $\partial_\sigma J^\sigma = 0$ for abelian theory, or $D_\sigma J^\sigma = 0$ for non-abelian theory, and to have the result be as uniquely-determined as possible. Based on the development in sections 6 and 7, we have four choices of solution: a) the massive non-abelian solution $G_\mu = \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ of (6.27); b) the massive abelian solution $G_\mu = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ of (6.28) which is simply solution (a) with $V = 0$; c) the massless non-abelian solution $G_\mu = \left(-V + k_\tau k^\tau + i\varepsilon\right)^{-1} J_\mu$ of (7.26) in the $k_\tau G^\tau = 0$ gauge which is simply solution (a) with $m = 0$; and d) the massless abelian solution $G_\mu = \left(k_\tau k^\tau + i\varepsilon\right)^{-1} J_\mu$ of (7.27) which is simply solution (b) with $m = 0$ or solution (c) with $V = 0$. Because one can follow Coleman-Zee as shown in sections 6 and 7 to include a massive boson solution $m \neq 0$ and then arrive at the massless solution simply by setting $m = 0$, and because the massless solution is uniquely forced to the $\xi = 1$ gauge to preserve continuity and thus we arrive at the exact same point whether we start with a massive or a massless solution, it makes more sense to first include the mass $m \neq 0$. This is a more general approach, and as we have seen, this mass can always be zeroed out later at the appropriate time, whereby the requirement for continuity will contextually fix the gauge into the Feynman / continuity gauge $\xi = 1$.

But there is also another more specific reason for starting with $m \neq 0$ beyond its generality, and that has specifically to do with the *uniqueness* of the massive solutions. Even though the continuity relationships $D_\sigma J^\sigma = 0$ and $\partial_\sigma J^\sigma = 0$ do zero out the terms containing the gauge number ξ from the massless bosons and contextually fix the gauge to $\xi = 1$, see (7.22) and (7.23), the condition $k_\tau G^\tau = 0$ is *required* for a massive boson but is simply a covariant choice of gauge condition for a massless gauge boson. So if we start with massive solution (a) which is $G_\mu = \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$, we know that the gauge condition $k_\tau G^\tau = 0$ *must* be in place because that is a requirement to ensure continuity for the massive solution, and that the perturbation V appears in simple form in this solution precisely because $k_\tau G^\tau = 0$, see (6.6) and (6.7), and (6.24). On the other hand, if we start with massless solution (c) which is $G_\mu = \left(-V + k_\tau k^\tau + i\varepsilon\right)^{-1} J_\mu$, we know even though the gauge number is contextually fixed to $\xi = 1$ by continuity, again, (7.22) and (7.23), that $k_\tau G^\tau = 0$ is merely a *choice* of gauge, and that the manner in which the perturbation V appears in $G_\mu = \left(-V + k_\tau k^\tau + i\varepsilon\right)^{-1} J_\mu$ is itself dependent upon this choice of $k_\tau G^\tau = 0$ gauge. If we choose $k_\tau G^\tau \neq 0$, then $G_\mu = \left(-V + k_\tau k^\tau + i\varepsilon\right)^{-1} J_\mu$ will have to include this $k_\tau G^\tau \neq 0$, and so its very form will change. So solution (a) is *uniquely* determined in all respects up to the covariant gauge condition $D_\nu D^\nu \theta = 0$ a.k.a. $\partial_\nu \partial^\nu \theta - i\partial_\nu \left[G^\nu, \theta\right] = 0$ developed after (6.5), while solution (c) is contextually fixed to the $\xi = 1$ gauge by continuity but $D_\nu G^\nu = \partial_\nu G^\nu$ remains a free scalar object which is *not required to be zero* and so renders the massless solutions weaker, i.e., less-unique than the massive solutions. Again, this solution will only be $G_\mu = \left(-V + k_\tau k^\tau + i\varepsilon\right)^{-1} J_\mu$ if we choose $k_\tau G^\tau = 0$ and *will* *change in form* in the event we choose a $k_\tau G^\tau \neq 0$ whereby we will explicitly have to include a $k_\tau G^\tau$ term.

So to preserve generality and maximize uniqueness, we shall now use solution (a), namely $G_\mu = \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ of (6.27) to replace each occurrence of G_μ with $\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ in (9.1). This has a *required* gauge relation $k_\tau G^\tau = 0$, and a selected gauge condition $D_\nu D^\nu \theta = 0$ which does not change the form of the solution in the event one chooses $D_\nu D^\nu \theta \neq 0$, see (6.5) and thereafter. As noted, this becomes solution (b) if we set V=0, this becomes solution (c) if we set m=0 *and choose* $k_\nu G^\nu = 0$ *as a gauge condition*, and it becomes solution (d) if we set V=0 and m=0 and again choose $k_\nu G^\nu = 0$. Thus, inserting $G_\mu = \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ into (9.1) we obtain:

$$
\begin{aligned}
\oiint F &= \iiint P' = -i\iiint dGG = -i\iiint d\left[G,G\right] = -i\oiint\left[G,G\right] \\
&= \oiint \tfrac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = \iiint \tfrac{1}{3!} P'_{\sigma\mu\nu} dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\iiint \tfrac{1}{3!} \begin{pmatrix} \partial_{[\sigma}\left(\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\mu]}\right)\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\nu \\ +\partial_{[\mu}\left(\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\nu]}\right)\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\sigma \\ +\partial_{[\nu}\left(\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\sigma]}\right)\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu \end{pmatrix} dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\iiint \tfrac{1}{3!} \begin{pmatrix} \partial_{\sigma}\left[\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\mu},\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\nu\right] \\ +\partial_{\mu}\left[\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\nu},\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\sigma\right] \\ +\partial_{\nu}\left[\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\sigma},\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu\right] \end{pmatrix} dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -i\oiint \tfrac{1}{2!}\left[\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_{\mu},\left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\nu\right] dx^\mu \wedge dx^\nu \neq 0
\end{aligned}
\quad .(9.2)
$$

This is the complete expression for the *net*-flux $\oiint F$ of the non-abelian magnetic field over a *closed* two-dimensional surface, and as we just learned in section 8, it is highly nonlinear, and indeed, contains an infinite recursion of $G_\tau\left(G_\tau, J_\tau\right)$ which is ultimately made into $G_\tau\left(J_\tau\right)$ by recursing to infinity then setting $V = 0$ as shown in (8.20). Indeed, we could also have employed $G_\mu = \pi_\infty J_\mu$ in from (8.20) in (9.1) to alternatively and equivalently obtain:

$$\begin{aligned}\oiint F\left(\left(0\right)\right)_{\infty} &= \iiint P'\left(\left(0\right)\right)_{\infty} = -i\iiint dGG\left(\left(0\right)\right)_{\infty} = -i\iiint d\left[G,G\right]\left(\left(0\right)\right)_{\infty} = -i\oiint\left[G,G\right]\left(\left(0\right)\right)_{\infty} \\
&= \oiint \tfrac{1}{2!}F_{\mu\nu}\left(\left(0\right)\right)_{\infty} dx^{\mu}\wedge dx^{\nu} = \iiint \tfrac{1}{3!}P'_{\sigma\mu\nu}\left(\left(0\right)\right)_{\infty} dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= -i\iiint \tfrac{1}{3!}\left(\partial_{[\sigma}\pi_{\infty}J_{\mu]}\pi_{\infty}J_{\nu} + \partial_{[\mu}\pi_{\infty}J_{\nu]}\pi_{\infty}J_{\sigma} + \partial_{[\nu}\pi_{\infty}J_{\sigma]}\pi_{\infty}J_{\mu}\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= -i\iiint \tfrac{1}{3!}\left(\partial_{\sigma}\left[\pi_{\infty}J_{\mu},\pi_{\infty}J_{\nu}\right] + \partial_{\mu}\left[\pi_{\infty}J_{\nu},\pi_{\infty}J_{\sigma}\right] + \partial_{\nu}\left[\pi_{\infty}J_{\sigma},\pi_{\infty}J_{\mu}\right]\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= -i\oiint \tfrac{1}{2!}\left[\pi_{\infty}J_{\mu},\pi_{\infty}J_{\nu}\right]dx^{\mu}\wedge dx^{\nu} \neq 0
\end{aligned} \quad .(9.3)$$

We will eventually return at the end of section 10 to discuss (9.3) above in more detail. But at the moment, (9.2) is in a form that better facilities understanding the connection between P' and a baryon density, because we can set $V=0$ at any order n of recursion we choose and thereby obtain $\oiint F\left(\left(0\right)\right)_{n}$.

Before trying to tackle the highly-nonlinear (9.2), see the section 8 discussion of recursion that is inherent in the above because (9.2) contains the perturbation $-V = k_{\tau}G^{\tau} + G_{\tau}k^{\tau} + G_{\tau}G^{\tau}$ of (6.7) throughout, let us now do what is commonly done in many other situations in particle physics: consider the zero-perturbation limit by setting V=0 throughout (9.2) right away. That is, we obtain and explore $\oiint F\left(\left(0\right)\right)_{0}$. This will of course remove the non-linear physics occurring in (9.2), but it will readily reveal why these faux magnetic monopoles have the symmetries that one expects to see in a baryon. Moreover, surprisingly enough, when we use $\oiint F\left(\left(0\right)\right)_{0}$ to calculate the energies associated with the flux equation $\iiint P' = -i\oiint\left[G,G\right]$ after some development of the baryon into protons and neutrons, we find a surprising, very tight concurrence with the binding energies that are experimentally-observed in nuclear physics, which suggests that the nuclear binding energies are in fact expressive of the behaviors of (9.2) in this zero-perturbation limit, i.e., in the linear / abelian approximation (see [15] sections 6 through 12 and all of [16]).

Once we set V=0 in each of the $\left(-V + k_{\tau}k^{\tau} - m^{2} + i\varepsilon\right)^{-1}$ in (9.2), these each become the ordinary denominator $1/\left(k_{\tau}k^{\tau} - m^{2} + i\varepsilon\right)$, because as developed in (6.26), it is $-V_{AB} = k_{\sigma}G^{\sigma}{}_{AB} + G_{\sigma AB}k^{\sigma} + \left(G_{\sigma}G^{\sigma}\right)_{AB}$ which is responsible for our having to write (9.2) with inverses rather than denominators. Thus, setting V=0 and rearranging somewhat, (9.2) for $\oiint F\left(\left(0\right)\right)_{0}$ and $P'\left(\left(0\right)\right)_{0}$ becomes:

$$\oiint F\left(\left(0\right)\right)_0 = \iiint P'\left(\left(0\right)\right)_0 = -i\iiint dGG\left(\left(0\right)\right)_0 = -i\iiint d\left[G,G\right]\left(\left(0\right)\right)_0 = -i\oiint\left[G,G\right]\left(\left(0\right)\right)_0$$
$$= \oiint \tfrac{1}{2!}F_{\mu\nu}\left(\left(0\right)\right)_0 dx^\mu \wedge dx^\nu = \iiint \tfrac{1}{3!}P'_{\sigma\mu\nu}\left(\left(0\right)\right)_0 dx^\sigma \wedge dx^\mu \wedge dx^\nu$$
$$= -i\iiint \tfrac{1}{3!}\left(\frac{\partial_{[\sigma}J_{\mu]}J_\nu}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial_{[\mu}J_{\nu]}J_\sigma}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial_{[\nu}J_{\sigma]}J_\mu}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right)dx^\sigma \wedge dx^\mu \wedge dx^\nu \,. \quad (9.4)$$
$$= -i\iiint \tfrac{1}{3!}\left(\frac{\partial_\sigma\left[J_\mu, J_\nu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial_\mu\left[J_\nu, J_\sigma\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial_\nu\left[J_\sigma, J_\mu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right)dx^\sigma \wedge dx^\mu \wedge dx^\nu$$
$$= -i\oiint \tfrac{1}{2!}\frac{\left[J_\mu, J_\nu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}dx^\mu \wedge dx^\nu \neq 0$$

Although the complete non-linear physics of $\oiint F \neq 0$ is described by (9.2) and alternatively (9.3), the simplified (9.4) enables us to reveal certain key symmetries for $\oiint F \neq 0$ which will support the view that the faux magnetic monopole density P' is in fact a baryon density, which symmetries carry over fully to the more-complete, highly-perturbed (9.2), (9.3). We shall refer to (9.4) as the "ground state" monopole equation, because the perturbations are zeroed out immediately before any levels of recursion are carried out.

Of particular interest, let us now focus on the $= -i\iiint d\left[G,G\right]\left(\left(0\right)\right)_0$ term in (9.4), which we restructure into:

$$\oiint F\left(\left(0\right)\right)_0 = \oiint \tfrac{1}{2!}F_{\mu\nu}\left(0\right)dx^\mu \wedge dx^\nu$$
$$= \iiint P'\left(\left(0\right)\right)_0 = \iiint \tfrac{1}{3!}P'_{\sigma\mu\nu}\left(\left(0\right)\right)_0 dx^\sigma \wedge dx^\mu \wedge dx^\nu = -i\iiint d\left[G,G\right]\left(\left(0\right)\right)_0 \quad . \quad (9.5)$$
$$= -i\iiint \tfrac{1}{3!}\left(\frac{\partial_\sigma\left[J_\mu, J_\nu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial_\mu\left[J_\nu, J_\sigma\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial_\nu\left[J_\sigma, J_\mu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right)dx^\sigma \wedge dx^\mu \wedge dx^\nu$$

From this we extract the faux magnetic monopole density raised to contravariant indexes:

$$P'^{\sigma\mu\nu}\left(\left(0\right)\right)_0 = -i\left(\frac{\partial^\sigma\left[J^\mu, J^\nu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial^\mu\left[J^\nu, J^\sigma\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} + \frac{\partial^\nu\left[J^\sigma, J^\mu\right]}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2}\right). \quad (9.6)$$

Now we take the crucial step of developing the current sources densities J^μ in terms of the underlying fermion wavefunctions ψ which arise in Dirac theory. Specifically, in abelian gauge theory, Dirac's equation says that $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$. For the adjoint spinor $\overline{\psi} = \psi^\dagger\gamma^0$ the field equation is $i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0$. Adding yields $\partial_\mu\left(\overline{\psi}\gamma^\mu\psi\right) = 0$ as is well known. And because the conserved current is expressed by $\partial_\mu J^\mu = 0$, we identify the current density with $J^\mu = \overline{\psi}\gamma^\mu\psi$, where each Dirac wavefunction ψ in a U(1) theory is of course a four-component column vector.

In non-abelian gauge theory, for the compact simple gauge group SU(N) (or for the product group SU(N)xU(1) with a U(1) factor that is required for magnetic monopoles to be topological stability as will be reviewed in section 10), the generalized wavefunction $\Psi = \Psi_A$, $A = 1...N$ is an Nx4 column vector of 4-component Dirac wavefunctions ψ. This non-abelian wavefunction Ψ may then subsist in any one of N distinct eigenstates. For example, for the $SU(3)_C$ group of chromodynamic strong interactions, the three (3) eigenstates are generally denoted (R)ed, (G)reen, (B)lue, and these distinct eigenstates are used to enable a baryon containing three quarks to satisfy the Fermi-Dirac-Pauli Exclusion Principle. Explicitly defined, using the SU(N) group generators $\lambda^i = \lambda^i_{AB}$, $i = 1...N^2 - 1$, the current density generalizes to $J^\mu = \lambda^i_{AB} J^{i\mu} = \lambda^i_{AB}\overline{\Psi}_C\lambda^i_{CD}\gamma^\mu\Psi_D \equiv \overline{\Psi}\gamma^\mu\Psi$, with Yang-Mills adjoint i and fundamental A,B,C,D indexes explicitly shown for illustration, and where as already stated $\Psi = \Psi_A$ is an N-component column vector of N fermion eigenstates. As has been reviewed at length earlier staring at (5.20), this current density satisfies the continuity relationship $D_\nu J^\nu = 0$. For SU(N)xU(1), we may for simplicity use λ^i_{AB} with $i = 0...N^2 - 1$, where we denote the U(1) generator as λ^0_{AB} with the "0" index. If we suppress the A,B,C,D indexes, then $J^\mu = \lambda^i J^{i\mu} = \lambda^i\overline{\Psi}\lambda^i\gamma^\mu\Psi \equiv \overline{\Psi}\gamma^\mu\Psi$.

So now, into (9.6), we first substitute $J^\mu = \lambda^i J^{i\mu}$, then $J^{i\mu} = \overline{\Psi}\lambda^i\gamma^\mu\Psi$, and then use $\left[\lambda^i, \lambda^j\right]\left(\overline{\Psi}\lambda^i\gamma^\mu\Psi\right)\left(\overline{\Psi}\lambda^j\gamma^\nu\Psi\right) = \left[\overline{\Psi}\gamma^\mu\Psi, \overline{\Psi}\gamma^\nu\Psi\right]$ (just a variant of $\left[\lambda^i, \lambda^j\right]A^{i\mu}B^{j\nu} = \left[A^\mu, B^\nu\right]$) in (9.6) to "populate" the faux Yang-Mills magnetic monopole with fermions. The result is:

$$P'^{\sigma\mu\nu}\left((0)\right)_0=-i\left(\frac{\partial^\sigma\left(\left[\lambda^i,\lambda^j\right]J^{i\mu}J^{j\nu}\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\mu\left(\left[\lambda^i,\lambda^j\right]J^{i\nu}J^{j\sigma}\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\nu\left(\left[\lambda^i,\lambda^j\right]J^{i\sigma}J^{j\mu}\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\right)$$

$$=-i\left(\begin{array}{l}\dfrac{\partial^\sigma\left(\left[\lambda^i,\lambda^j\right]\left(\overline{\Psi}\lambda^i\gamma^\mu\Psi\right)\left(\overline{\Psi}\lambda^j\gamma^\nu\Psi\right)\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\\+\dfrac{\partial^\mu\left(\left[\lambda^i,\lambda^j\right]\left(\overline{\Psi}\lambda^i\gamma^\nu\Psi\right)\left(\overline{\Psi}\lambda^j\gamma^\nu\Psi\right)\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\\+\dfrac{\partial^\nu\left(\left[\lambda^i,\lambda^j\right]\left(\overline{\Psi}\lambda^i\gamma^\sigma\Psi\right)\left(\overline{\Psi}\lambda^j\gamma^\mu\Psi\right)\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\end{array}\right) \qquad . \qquad (9.7)$$

$$=-i\left(\frac{\partial^\sigma\left[\overline{\Psi}\gamma^\mu\Psi,\overline{\Psi}\gamma^\nu\Psi\right]}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\mu\left[\overline{\Psi}\gamma^\nu\Psi,\overline{\Psi}\gamma^\sigma\Psi\right]}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\nu\left[\overline{\Psi}\gamma^\sigma\Psi,\overline{\Psi}\gamma^\mu\Psi\right]}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\right)$$

We could just as readily have just inserted $J^\mu=\overline{\Psi}\gamma^\mu\Psi$ into (9.6) to arrive directly at the bottom line of (9.7), but it is helpful to see the intermediate calculations which explicitly contain the group generators. Given that $\oiint F=\iiint P'$, and referring back to the discussion at the end of section 3, we now see for the first time the manner in which $\oiint F\left(G\left(J\left(\psi\right)\right)\right)$, that is, the manner in which the *composite* faux magnetic monopole $\oiint F$ arising from the faux magnetic source $P'=-idGG=-id\left[G,G\right]$ does indeed contain fermion wavefunctions Ψ. Now, we shall show how these fermion wavefunction in fact possess all of the key symmetries required to qualify them as colored quarks, and how $P'^{\sigma\mu\nu}$ possesses all of the key symmetries of a baryon.

The first thing we observe is that $P'^{\sigma\mu\nu}\left((0)\right)_0$ contains three additive terms. And, as discussed moments ago, for SU(N) or for SU(N)xU(1), each $\Psi=\Psi_A$ is an N-component column vector of 4-component Dirac wavefunctions ψ which may subsist in any one of N distinct eigenstates. So if we regard $P'^{\sigma\mu\nu}\left((0)\right)_0$ as a composite system of more than one fermion, then each fermion in this system must be placed into a distinct eigenstate in order to satisfy the Fermion Exclusion Principle. The three additive terms in (9.7) advise us that there are a total of three such fermion eigenstates which constitute $P'^{\sigma\mu\nu}\left((0)\right)_0$, and so we label these eigenstates among the three additive terms as Ψ_1,Ψ_2,Ψ_3. With this we now rewrite (9.7), including a restructuring $\left[\overline{\Psi}\gamma^\mu\Psi,\overline{\Psi}\gamma^\nu\Psi\right]=\overline{\Psi}\gamma^{[\mu}\Psi\overline{\Psi}\gamma^{\nu]}\Psi$ of the commutators in the bottom line below, as:

$$
\begin{aligned}
&P'^{\sigma\mu\nu}\left((0)\right)_0 \\
&=-i\left(\frac{\partial^\sigma\left[\overline{\Psi}_1\gamma^\mu\Psi_1,\overline{\Psi}_1\gamma^\nu\Psi_1\right]}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\mu\left[\overline{\Psi}_2\gamma^\nu\Psi_2,\overline{\Psi}_2\gamma^\sigma\Psi_2\right]}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\nu\left[\overline{\Psi}_3\gamma^\sigma\Psi_3,\overline{\Psi}_3\gamma^\mu\Psi_3\right]}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\right). \\
&=-i\left(\frac{\partial^\sigma\left(\overline{\Psi}_1\gamma^{[\mu}\Psi_1\overline{\Psi}_1\gamma^{\nu]}\Psi_1\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\mu\left(\overline{\Psi}_2\gamma^{[\nu}\Psi_2\overline{\Psi}_2\gamma^{\sigma]}\Psi_2\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}+\frac{\partial^\nu\left(\overline{\Psi}_3\gamma^{[\sigma}\Psi_3\overline{\Psi}_3\gamma^{\mu]}\Psi_3\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\right)
\end{aligned}
\tag{9.8}
$$

Because we must be able to place the fermions into one of three distinct eigenstates in order to satisfy Exclusion for the composite ground state faux monopole $P'^{\sigma\mu\nu}\left((0)\right)_0$, we must now chose a rank-3 gauge group in order to enforce this exclusion. There are two apparent choices. First is the simple group SU(3). Second is the product group SU(3)×U(1). But as we shall see in the next section, there really is not a choice and we actually must choose SU(3)×U(1). But to start simply, let us *assume* the simpler choice of SU(3) until contradicted, and then see why we are later compelled by contradiction to amend this choice to SU(3)×U(1). Choosing SU(3), we first label eigenstates. Because the labels are arbitrary, we use the names of some colors, say, (R)ed, (G)reen, (B)lue. Thus, using the SU(3) generators λ^i normalized to $\mathrm{Tr}\left(\lambda^i\right)^2=\frac{1}{2}$ we define:

$$
\Psi_1\equiv\left|\lambda^8=\tfrac{1}{\sqrt{3}};\lambda^3=0\right\rangle=\begin{pmatrix}\psi_R\\0\\0\end{pmatrix};\Psi_2\equiv\left|\lambda^8=-\tfrac{1}{2\sqrt{3}};\lambda^3=\tfrac{1}{2}\right\rangle=\begin{pmatrix}0\\\psi_G\\0\end{pmatrix};\Psi_3\equiv\left|\lambda^8=-\tfrac{1}{2\sqrt{3}};\lambda^3=-\tfrac{1}{2}\right\rangle=\begin{pmatrix}0\\0\\\psi_B\end{pmatrix}. \tag{9.9}
$$

Now, all of a sudden, in a very consequential step, we see how these $P'^{\sigma\mu\nu}\left((0)\right)_0$ ground state magnetic monopole densities contain three fermions in one of three eigenstates R, G, B, and how SU(3) (or really, SU(3)×U(1) as we shall see in the next section) emerges as a *required* gauge group in order to force exclusion upon the fermions that comprise $P'^{\sigma\mu\nu}\left((0)\right)_0$. In other words, we have never had to *postulate* SU(3) *per se* in order to force exclusion on the quarks within *experimentally*-observed baryons. Rather, *we have been forced to introduce SU(3)* (or at least a rank-3 gauge group) in order to ensure proper Exclusion for the fermions of the *theoretically*-motivated $P'^{\sigma\mu\nu}$ which first emerged back in (3.3) when we found that $\oiint F\neq 0$ in a non-abelian gauge theory, and when we found that the underlying magnetic charge density was the composite $P'=-idGG=-id\left[G,G\right]$ which is faux-assembled from the gauge fields G. At the same time, because we are *required* to select a rank-3 gauge group which for now is SU(3), and because we have labelled the eigenstates with the names of colors, there are now eight gauge bosons $G^i{}_\mu$ in $G_\mu=\lambda^i G^i{}_\mu$ associated with (9.8), and each of these will be bi-colored, just as are the gluons of chromodynamic theory. This means that we may be able to obviate the need for a separate postulation of classical or quantum chromodynamics, such that *chromodynamics no longer a fundamental theory, but rather is a corollary, secondary theory* that emerges in the process of enforcing fermion Exclusion upon the fermions contained in the non-abelian faux magnetic monopole density (9.8).

Now we focus on the terms of the form $\Psi\overline{\Psi}$ which appear in the bottom line of (9.8). These terms have a column vector to the left of a row vector, and using (9.9), these may be explicitly written in 3x3 matrix form as:

$$\Psi_1\overline{\Psi_1}=\begin{pmatrix}\psi_R\overline{\psi_R} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};\quad \Psi_2\overline{\Psi_2}=\begin{pmatrix}0 & 0 & 0\\ 0 & \psi_G\overline{\psi_G} & 0\\ 0 & 0 & 0\end{pmatrix};\quad \Psi_3\overline{\Psi_3}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \psi_B\overline{\psi_B}\end{pmatrix}. \tag{9.10}$$

We may then use this to rewrite (9.8) in explicit 3x3 matrix form:

$$P'^{\sigma\mu\nu}\left((0)\right)_0=-i\begin{pmatrix}\frac{\partial^\sigma\left(\overline{\Psi_1}\gamma^{[\mu}\psi_R\overline{\psi_R}\gamma^{\nu]}\Psi_1\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2} & 0 & 0\\ 0 & \frac{\partial^\mu\left(\overline{\Psi_2}\gamma^{[\nu}\psi_G\overline{\psi_G}\gamma^{\sigma]}\Psi_2\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2} & 0\\ 0 & 0 & \frac{\partial^\nu\left(\overline{\Psi_3}\gamma^{[\sigma}\psi_B\overline{\psi_B}\gamma^{\mu]}\Psi_3\right)}{\left(k_\tau k^\tau-m^2+i\varepsilon\right)^2}\end{pmatrix}. \tag{9.11}$$

Next, we focus in on $\psi_R\overline{\psi_R}=u_R\overline{u_R}$, $\psi_G\overline{\psi_G}=u_G\overline{u_G}$ and $\psi_B\overline{\psi_B}=u_B\overline{u_B}$ which involve ordinary, four-component Dirac wavefunctions ψ and spinors u, and we focus especially on the $u\overline{u}$ which contain a column spinor to the left of a row spinor. Often, the Dirac spin sum relationship is normalized to $N^2=\mathrm{E}+m$ and so is written as $\Sigma_{\text{spins}}u\overline{u}=\left(\not{p}+m\right)$. But if we wish to be more general and defer a decision on normalization, we may employ in (9.11) the spin sum *prior to normalization*, which is (see, e.g., [14] exercise 5.9):

$$\sum\nolimits_{\text{spins}}u\overline{u}=\frac{N^2}{E+m}\left(\not{p}+m\right). \tag{9.12}$$

So, if we now take the sum over all spins $\Sigma_{\text{spins}}P'^{\sigma\mu\nu}\left((0)\right)_0$ of the faux monopole (9.11), and if we apply (9.12) in the form $\Sigma_{\text{spins}}u_C\overline{u_C}=N^2\left(\not{p}_C+m_C\right)/\left(E_C+m_C\right)$ to each color $C=R,G,B$ of fermion, we may use (9.12) to rewrite (9.11), for the moment without $+i\varepsilon$, as:

$$\sum_{\text{spins}} P'^{\sigma\mu\nu}\left((0)\right)_0 =$$

$$-i\begin{pmatrix} \frac{N^2}{E_R+m_R}\frac{\partial^\sigma\left(\overline{\Psi}_1\gamma^{[\mu}\left(\not{p}_R+m_R\right)\gamma^{\nu]}\Psi_1\right)}{\left(k_\tau k^\tau-m^2\right)^2} & 0 & 0 \\ 0 & \frac{N^2}{E_G+m_G}\frac{\partial^\mu\left(\overline{\Psi}_2\gamma^{[\nu}\left(\not{p}_G+m_G\right)\gamma^{\sigma]}\Psi_2\right)}{\left(k_\tau k^\tau-m^2\right)^2} & 0 \\ 0 & 0 & \frac{N^2}{E_B+m_B}\frac{\partial^\nu\left(\overline{\Psi}_3\gamma^{[\sigma}\left(\not{p}_B+m_B\right)\gamma^{\mu]}\Psi_3\right)}{\left(k_\tau k^\tau-m^2\right)^2} \end{pmatrix}. \quad (9.13)$$

Next we next turn our attention to the expressions $\left(\not{p}_C+m_C\right)/\left(k_\tau k^\tau-m^2\right)$ which appear in each diagonal entry above. We simultaneously take note of the fact that the fermion propagator $i\left(\not{p}-m\right)^{-1}$ sans $+i\varepsilon$ is related by a constant factor *i* to:

$$\frac{\not{p}+m}{p^\tau p_\tau-m^2}=\frac{\not{p}+m}{\left(\not{p}+m\right)\left(\not{p}-m\right)}=\left(\not{p}-m\right)^{-1}. \quad (9.14)$$

So we are motivated to see if there is a basis upon which we may set the $\left(\not{p}_C+m_C\right)/\left(k_\tau k^\tau-m^2\right)$ terms in (9.13) to $\left(\not{p}-m\right)^{-1}$ and thereby introduce the propagator for each of these fermions directly into (9.13). For this, we return to the discussion of sections 6 and 7 during which we developed inverse solutions to the electric charge equation $-J^\nu=\left(g^{\nu\sigma}D_\tau D^\tau-D^\sigma D^\nu\right)G_\sigma$ in both massive and massless form, and where we also reviewed the degrees of freedom of various solutions and related questions of uniqueness.

Each term in equation (9.13) contains $1/\left(k_\tau k^\tau-m^2\right)^2$, that is $1/\left(k_\tau k^\tau-m^2\right)$ times itself. As noted in the mass shell discussion prior to (6.24), we are using p^τ and k^σ respectively to denote fermion and boson momentum vectors. And, of course, each $1/\left(k_\tau k^\tau-m^2\right)$ entered (9.13) back at (9.2) when we inserted the massive boson inverse solution $G_\mu=\left(-V+k_\tau k^\tau-m^2+i\varepsilon\right)^{-1}J_\mu$ of (6.27) into (9.1). As reviewed in sections 6 and 7, this solution, in view of the continuity requirement $D_\nu J^\nu=0$ of (5.20) and the consequently-mandated covariant gauge $D_\nu G^\nu=0$ of (6.5) is *unique* up to the gauge condition $D_\nu D^\nu\theta=0$ a.k.a. $\partial_\nu\partial^\nu\theta-i\partial_\nu\left[G^\nu,\theta\right]=0$. And this solution is unchanged in form under a non-abelian gauge transformation because nowhere does the unphysical parameter θ appear in any of the covariant physics equations. So in trying to match up $\left(\not{p}_C+m_C\right)/\left(k_\tau k^\tau-m^2\right)$ which appears in (9.13) with $\left(\not{p}+m\right)/\left(p^\tau p_\tau-m^2\right)$ in the fermion propagator-related (9.14), we see that the numerators match up perfectly but there is a mismatch in the denominators. Particularly, each $k_\tau k^\tau-m^2$ in (9.13) is the propagator denominator for a massive gauge boson which has *three* degrees of freedom, while $p^\tau p_\tau - m^2$ in (9.14) is the propagator denominator for a massive fermion which has *four* degrees of freedom. So, how do we match these up, and what impact, if any, might this have on the uniqueness of the solution $G_\mu = \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ upon which (9.13) is based?

Because each of the boson propagator denominators $1/\left(k_\tau k^\tau - m^2\right)$ in (9.13) represents a massive boson with three degrees of freedom, the term $1/\left(k_\tau k^\tau - m^2\right)^2$ which is a product of two boson propagator denominators thus represents six degrees of freedom. So we now take each $1/\left(k_\tau k^\tau - m^2\right)\left(k_\tau k^\tau - m^2\right)$ and shift one degree of freedom from the first $1/\left(k_\tau k^\tau - m^2\right)$ into the second $1/\left(k_\tau k^\tau - m^2\right)$. That is, keeping in mind that p^τ and k^σ respectively denote fermion and boson momentum vectors and that the former has four degrees of freedom (particle / antiparticle in each of spin up and spin down states) and the latter when massive has three degrees of freedom (two transverse polarizations, one longitudinal), we rewrite $1/\left(k_\tau k^\tau - m^2\right)^2$ as:

$$\frac{1}{\left(k_\tau k^\tau - m^2\right)^2} = \frac{1}{\left(k_\tau k^\tau - m^2\right)\left(k_\tau k^\tau - m^2\right)} = \frac{1}{k_\tau k^\tau \left(p_\tau p^\tau - m^2\right)}. \tag{9.15}$$

What we have effectively done is to take the 6=3+3 degrees of freedom represented in the first term, and redistribute them into 6=2+4 degrees of freedom represented in the final term. In the final term, therefore, we have turned one originally-massive gauge boson propagator denominator $1/\left(k_\tau k^\tau - m^2\right)$ into a massless gauge boson propagator denominator $1/k_\tau k^\tau$. But at the same time, we have turned the other originally-massive gauge boson propagator denominator $1/\left(k_\tau k^\tau - m^2\right)$ into a massive fermion propagator denominator $1/\left(p_\tau p^\tau - m^2\right)$. This is very analogous to the Goldstone mechanism used to give mass to massless gauge bosons by shifting a degree of freedom from a scalar field into a boson field. *Here, we are simply shifting a degree of freedom from a boson field into a fermion field.*

Now we saw of course in sections 6 and 7 that the solution for a massless gauge boson was less-unique than that for a massive boson, precisely because the massless gauge boson has one less degree of freedom. But we also saw how context matters, and how the context of a conserved current $D_\nu J^\nu = 0$ contextually fixed the massless boson into the Feynman / continuity gauge $\xi = 1$. The only *contextual* loss of uniqueness in the massless solution, therefore, was that $D_\nu G^\nu = 0$ was no longer a mandatory constraint but instead was relegated to a mere choice of gauge, which meant that $\partial_\nu G^\nu = 0$ was also demoted from a requirement of continuity to an optional gauge condition. And all of the non-uniqueness of the massless solution, even before the application of continuity $D_\nu J^\nu = 0$ fixed the gauge number to $\xi = 1$, emanated from *removing* a degree of freedom when going from a massive to a massless gauge boson. But in (9.15) we are not *removing* any degrees of freedom as we did in going from section 6 to section

7. We are merely shifting them around in the overall context of (9.13) according to the recipe of (9.15). So after we apply (9.15) to (9.13), we will not in any way alter the uniqueness of (9.13). It will remain just as uniquely-specified after (9.15), as before (9.15). Effectively, we *contextually* dodge the additional non-uniqueness that emerges in going from the massive solutions of section 6 to the massless solutions of section 7, by *moving* rather than *removing* a degree of freedom, in the *context* of (9.13).

So let us now do exactly what we just said. We now use (9.15) in (9.13) to shift around the six degrees of freedom in each diagonal element from a 3+3 to a 2+4 configuration, and at the same time we label the p_τ and the m in relation to the color of the fermion in each term. Thus, without any loss of uniqueness, simply by shifting a degree of freedom, (9.13) becomes:

$$\sum\nolimits_{\text{spins}} P'^{\sigma\mu\nu}\left((0)\right)_0 = -i\begin{pmatrix} \frac{N^2}{E_R+m_R}\frac{\partial^\sigma\left(\overline{\Psi}_1\gamma^{[\mu}\left(\not{p}_R+m_R\right)\gamma^{\nu]}\Psi_1\right)}{k_\tau k^\tau\left(p_{R\tau}p_R{}^\tau-m_R{}^2\right)} & 0 & 0 \\ 0 & \frac{N^2}{E_G+m_G}\frac{\partial^\mu\left(\overline{\Psi}_2\gamma^{[\nu}\left(\not{p}_G+m_G\right)\gamma^{\sigma]}\Psi_2\right)}{k_\tau k^\tau\left(p_{G\tau}p_G{}^\tau-m_G{}^2\right)} & 0 \\ 0 & 0 & \frac{N^2}{E_B+m_B}\frac{\partial^\nu\left(\overline{\Psi}_3\gamma^{[\sigma}\left(\not{p}_B+m_B\right)\gamma^{\mu]}\Psi_3\right)}{k_\tau k^\tau\left(p_{B\tau}p_B{}^\tau-m_B{}^2\right)} \end{pmatrix}. \tag{9.16}$$

Importantly, in the process of shifting degrees of freedom, the remaining boson propagator denominator in each term has become $1/k_\tau k^\tau$ which is the propagator for a *massless* gauge boson. So now, the eight bi-colored gauge bosons of the required $SU(3)_C$ group have become massless, at the same time the fermions have acquired mass since they have four degrees of freedom following application of (9.15). Because the eight bi-colored gluons of QCD are also massless, this means that the gauge bosons associated with (9.16) have now have three very important symmetries that match up with the gluons of QCD: 1) there are eight of them, 2) they are bi-colored, and 3) they are massless. Yet, because of using a Goldstone-like method for what is a variant of the contextual gauge shifting discussed in section 7, *no uniqueness has been lost*.

Now we return to the normalization which we deferred back at (9.12). Often, as noted, the chosen normalization is $N^2 = \mathrm{E}+m$. Let us instead, however, for each term in (9.16), choose to *include the* $k_\tau k^\tau$ *massless boson term in the normalization*. That is, for each term in (9.16) let us now normalize to:

$$N^2 \equiv \left(E_C+m_C\right)k_\tau k^\tau . \tag{9.17}$$

So, applying the normalization (9.17), and propagator expression (9.14) for each fermion color $C = R,G,B$, we reduce (9.16) to:

$$\sum_{\text{spins}} P'^{\sigma\mu\nu}\left((0)\right)_0 = -i\begin{pmatrix} \partial^\sigma\left(\overline{\Psi_1}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\Psi_1\right) & 0 & 0 \\ 0 & \partial^\mu\left(\overline{\Psi_2}\gamma^{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\sigma]}\Psi_2\right) & 0 \\ 0 & 0 & \partial^\nu\left(\overline{\Psi_3}\gamma^{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\mu]}\Psi_3\right) \end{pmatrix}. \tag{9.18}$$

Next we look closely at one of the terms above, say, the term $\overline{\Psi_1}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\Psi_1$ on the upper left. Making explicit use of (9.9), this term, is:

$$\overline{\Psi_1}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\Psi_1 = \begin{pmatrix}\overline{\psi_R} & 0 & 0\end{pmatrix}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\begin{pmatrix}\psi_R \\ 0 \\ 0\end{pmatrix} = \overline{\psi_R}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R. \tag{9.19}$$

A similar result obtains for the other two terms, which now allows us to rewrite (9.18) as:

$$\sum_{\text{spins}} P'^{\sigma\mu\nu}\left((0)\right)_0 = -i\begin{pmatrix} \partial^\sigma\left(\overline{\psi_R}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R\right) & 0 & 0 \\ 0 & \partial^\mu\left(\overline{\psi_G}\gamma^{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\sigma]}\psi_G\right) & 0 \\ 0 & 0 & \partial^\nu\left(\overline{\psi_B}\gamma^{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\mu]}\psi_B\right) \end{pmatrix}. \tag{9.20}$$

Any time we wish to calculate with the propagator terms $i\left(\not{p}-m\right)^{-1}$ and also include $+i\varepsilon$, we set these to $i\left(\not{p}-m\right)^{-1} = i\left(\not{p}+m\right)/\left(p_\sigma p^\sigma - m^2 + i\varepsilon\right)$.

Finally, in another important step that will lead us to topological stability, we take the trace of the above. This yields the fully-developed, spin-summed trace of the faux monopole density $P' = -idGG = -id\left[G,G\right]$ in the zero-recursion, zero-perturbation limit $\left((0)\right)_0$, namely:

$$\begin{aligned}&\mathrm{Tr}\sum_{\text{spins}} P'^{\sigma\mu\nu}\left((0)\right)_0 \\ &= -i\left(\partial^\sigma\left(\overline{\psi_R}\gamma^{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma^{\nu]}\psi_R\right) + \partial^\mu\left(\overline{\psi_G}\gamma^{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma^{\sigma]}\psi_G\right) + \partial^\nu\left(\overline{\psi_B}\gamma^{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma^{\mu]}\psi_B\right)\right)\end{aligned}. \tag{9.21}$$

We shall now show how this has the identical symmetries as a baryon, how this leads directly to meson mediators of interactions between monopoles, how this requires us to choose SU(3)×U(1) rather than SU(3) as our rank-3 gauge group, how this leads to topological stability, and how the above becomes flavored into protons and neutrons.

10. Why the Composite Faux Magnetic Monopoles of Yang-Mills Gauge Theory have all of the Required Chromodynamic Symmetries of Baryons, and how these are Flavored into being Topologically-Stable Protons and Neutrons

In the trace form of (9.21), we see clearly that $\mathrm{Tr}\Sigma_{\mathrm{spins}} P'^{\sigma\mu\nu}\left((0)\right)_0$ is a third rank antisymmetric tensor in spacetime which will reverse sign under the interchange of any two adjacent indexes. From here, we simplify by just writing $\Sigma_{\mathrm{spins}} \to \Sigma$. Let us denote this fundamental antisymmetry, which is an inherent feature of any magnetic monopole in spacetime, using the wedge-product notation $\sigma \wedge \mu \wedge \nu$. If we now associate each color wavefunction with the spacetime index in the related ∂^σ operator in (9.21), i.e., $\sigma \sim R$, $\mu \sim G$ and $\nu \sim B$, and keeping in mind that $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left((0)\right)_0$ is antisymmetric in all spacetime indexes, we may use $\sigma \wedge \mu \wedge \nu \sim R \wedge G \wedge B = R[G,B] + G[B,R] + B[R,G]$ to express this antisymmetry. *But this is the exact colorless wavefunction that is expected of a baryon.* Indeed, *the antisymmetric character of the spacetime indexes in a magnetic monopole should have been a good tipoff that magnetic monopoles would naturally make good baryons*. So, we now may assert that the non-abelian composite faux monopole density $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left((0)\right)_0$ in the ground state (9.21) has the exact same antisymmetric colorless chromodynamic symmetry as does a baryon!

Now, let us lower the indexes in (9.21) and write this as the differential form relation:

$$
\begin{aligned}
&\mathrm{Tr}\Sigma P'\left((0)\right)_0 = \mathrm{Tr}\tfrac{1}{3!}\Sigma P'_{\sigma\mu\nu}\left((0)\right)_0 dx^\sigma \wedge dx^\mu \wedge dx^\nu \\
&= -\tfrac{1}{3!} i \begin{pmatrix} \partial_\sigma \left(\overline{\psi_R}\gamma_{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma_{\nu]}\psi_R\right) \\ +\partial_\mu \left(\overline{\psi_G}\gamma_{[\nu}\left(\not{p}_G - m_G\right)^{-1}\gamma_{\sigma]}\psi_G\right) \\ +\partial_\nu \left(\overline{\psi_B}\gamma_{[\sigma}\left(\not{p}_B - m_B\right)^{-1}\gamma_{\mu]}\psi_B\right) \end{pmatrix} dx^\sigma \wedge dx^\mu \wedge dx^\nu \; . \\
&= -\tfrac{1}{3!} i \partial_\sigma \begin{pmatrix} \overline{\psi_R}\gamma_{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma_{\nu]}\psi_R \\ +\overline{\psi_G}\gamma_{[\mu}\left(\not{p}_G - m_G\right)^{-1}\gamma_{\nu]}\psi_G \\ +\overline{\psi_B}\gamma_{[\mu}\left(\not{p}_B - m_B\right)^{-1}\gamma_{\nu]}\psi_B \end{pmatrix} dx^\sigma \wedge dx^\mu \wedge dx^\nu
\end{aligned}
\tag{10.1}
$$

In the bottom expression, a ∂_σ with the same σ index has been factored out of the entire expression. So now we can apply Gauss' / Stokes theorem to (10.1), and can use the forms in the top line of (9.1) to help us out.

Specifically, by expanding some of the forms in the top line of (9.1), we may write:

$$\begin{aligned}&\oiint F=\iiint P'=\oiint \tfrac{1}{2!}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}=\iiint \tfrac{1}{3!}P'_{\sigma\mu\nu}dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\&=-i\iiint d\left[G,G\right]=-i\iiint \tfrac{1}{3!}\left(\partial_{\sigma}\left[G_{\mu},G_{\nu}\right]+\partial_{\mu}\left[G_{\nu},G_{\sigma}\right]+\partial_{\nu}\left[G_{\sigma},G_{\mu}\right]\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\ .\\&=-i\oiint\left[G,G\right]=-i\oiint \tfrac{1}{2!}\left[G_{\mu},G_{\nu}\right]dx^{\mu}\wedge dx^{\nu}\end{aligned}\qquad(10.2)$$

Therefore, taking the zero perturbation limit $V=0$, summing all spins, taking the trace, and then injecting in the final expression from (10.1), we may write this as:

$$\begin{aligned}&\oiint \mathrm{Tr}\Sigma F\left(\left(0\right)\right)_{0}=\oiint \tfrac{1}{2!}\mathrm{Tr}\Sigma F_{\mu\nu}\left(\left(0\right)\right)_{0}dx^{\mu}\wedge dx^{\nu}=\iiint \mathrm{Tr}\Sigma P'\left(\left(0\right)\right)_{0}=\iiint \tfrac{1}{3!}\mathrm{Tr}\Sigma P'_{\sigma\mu\nu}\left(\left(0\right)\right)_{0}dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\&=-i\iiint \mathrm{Tr}\Sigma d\left[G,G\right]\left(\left(0\right)\right)_{0}=-i\iiint \tfrac{1}{3!}\mathrm{Tr}\Sigma\left(\partial_{\sigma}\left[G_{\mu},G_{\nu}\right]+\partial_{\mu}\left[G_{\nu},G_{\sigma}\right]+\partial_{\nu}\left[G_{\sigma},G_{\mu}\right]\right)\left(\left(0\right)\right)_{0}dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\&=-i\oiint \mathrm{Tr}\Sigma\left[G,G\right]\left(\left(0\right)\right)_{0}=-i\oiint \tfrac{1}{2!}\mathrm{Tr}\Sigma\left[G_{\mu},G_{\nu}\right]\left(\left(0\right)\right)_{0}dx^{\mu}\wedge dx^{\nu}\\&=-i\iiint \tfrac{1}{3!}\partial_{\sigma}\left(\overline{\psi_R}\gamma_{[\mu}\left(\not{p}_R-m_R\right)^{-1}\gamma_{\nu]}\psi_R+\overline{\psi_G}\gamma_{[\mu}\left(\not{p}_G-m_G\right)^{-1}\gamma_{\nu]}\psi_G+\overline{\psi_B}\gamma_{[\mu}\left(\not{p}_B-m_B\right)^{-1}\gamma_{\nu]}\psi_B\right)dx^{\sigma}\wedge dx^{\mu}\wedge dx^{\nu}\\&=-i\oiint \tfrac{1}{2!}\left(\overline{\psi_R}\gamma_{[\mu}\left(\not{p}_R-m_R\right)^{-1}\gamma_{\nu]}\psi_R+\overline{\psi_G}\gamma_{[\mu}\left(\not{p}_G-m_G\right)^{-1}\gamma_{\nu]}\psi_G+\overline{\psi_B}\gamma_{[\mu}\left(\not{p}_B-m_B\right)^{-1}\gamma_{\nu]}\psi_B\right)dx^{\mu}\wedge dx^{\nu}\end{aligned}\qquad.(10.3)$$

From this we extract several integrands with an overall multiplication by *i*:

$$\begin{aligned}&\mathrm{Tr}\Sigma iF_{\mathrm{eff}\,\mu\nu}\left(\left(0\right)\right)_{0}\equiv\mathrm{Tr}\Sigma\left[G_{\mu},G_{\nu}\right]\left(\left(0\right)\right)_{0}\\&=\overline{\psi_R}\gamma_{[\mu}\left(\not{p}_R-m_R\right)^{-1}\gamma_{\nu]}\psi_R+\overline{\psi_G}\gamma_{[\mu}\left(\not{p}_G-m_G\right)^{-1}\gamma_{\nu]}\psi_G+\overline{\psi_B}\gamma_{[\mu}\left(\not{p}_B-m_B\right)^{-1}\gamma_{\nu]}\psi_B\end{aligned}\ .\qquad(10.4)$$

This includes defining an "effective" $\mathrm{Tr}\Sigma iF_{\mathrm{eff}\,\mu\nu}\left(\left(0\right)\right)_{0}$. This is because while (1.5) tells us that $F_{\mu\nu}=\partial_{[\mu}G_{\nu]}-i\left[G_{\mu},G_{\nu}\right]$ so that $\mathrm{Tr}\Sigma\left[G_{\mu},G_{\nu}\right]=\mathrm{Tr}\Sigma iF_{\mu\nu}-\mathrm{Tr}\Sigma i\partial_{[\mu}G_{\nu]}$, as found in (3.5) the total net flux $\oiint F$ is invariant under the transformation $F^{\mu\nu}\rightarrow F^{\mu\nu}{}'=F^{\mu\nu}-\partial^{[\nu}G^{\mu]}$. This means that the gauge field is *not observable* with respect to net flux across closed surfaces of the monopole precisely because of the abelian subset expression $\oiint dG=\mathbf{0}$ which is responsible for there being no net flux of magnetic fields *at all* across a closed surface in abelian gauge theory. So while $\oiint \mathrm{Tr}\Sigma F\left(\left(0\right)\right)_{0}=-i\iiint \mathrm{Tr}\Sigma d\left[G,G\right]\left(\left(0\right)\right)_{0}$ in the integral formation of (10.3) by virtue of the symmetry principle (3.5), when the integrands are separately extracted as in (10.4), the actual relationship is $F_{\mu\nu}=\partial_{[\mu}G_{\nu]}-i\left[G_{\mu},G_{\nu}\right]$. But the *effective* relationship in terms of what actually becomes *net observable flux across closed surfaces*, is $F_{\mathrm{eff}\,\mu\nu}=-i\left[G_{\mu},G_{\nu}\right]$. That is the basis for the definition of $F_{\mathrm{eff}\,\mu\nu}$ in (10.4).

By inspection, $\mathrm{Tr}\Sigma\left[G_{\mu},G_{\nu}\right]\left(\left(0\right)\right)_{0}$ in (10.4) has the color wavefunction $\overline{R}R+\overline{G}G+\overline{B}B$ of a meson. But look at the context in which this meson wavefunction has appeared in (10.3): Using selected terms from (10.3), especially $\oiint \mathrm{Tr}\Sigma F\left(\left(0\right)\right)_{0}$, we see that:

$$\begin{aligned}&\oiint \mathrm{Tr}\Sigma F\left(\left(0\right)\right)_0 = -i\oiint \mathrm{Tr}\Sigma\left[G,G\right]\left(\left(0\right)\right)_0 = -i\oiint \tfrac{1}{2!}\mathrm{Tr}\Sigma\left[G_\mu,G_\nu\right]\left(\left(0\right)\right)_0 dx^\mu \wedge dx^\nu \\ &= -i\oiint \tfrac{1}{2!}\left(\overline{\psi_R}\gamma_{[\mu}\left(\not{p}_R - m_R\right)^{-1}\gamma_{\nu]}\psi_R + \overline{\psi_G}\gamma_{[\mu}\left(\not{p}_G - m_G\right)^{-1}\gamma_{\nu]}\psi_G + \overline{\psi_B}\gamma_{[\mu}\left(\not{p}_B - m_B\right)^{-1}\gamma_{\nu]}\psi_B\right)dx^\mu \wedge dx^\nu\end{aligned} \quad .(10.5)$$

So we see that the Yang-Mills magnetic fields which net-flow across closed surfaces of the composite, faux magnetic monopole density $P' = -idGG = -id\left[G,G\right]$ of non-abelian gauge theory in the form of $\oiint \mathrm{Tr}\Sigma F\left(\left(0\right)\right)_0$, have the $\overline{R}R + \overline{G}G + \overline{B}B$ color symmetry of mesons!

This is a very important finding. Back at (3.3) we identified a puzzle: We found that in non-abelian Yang-Mills gauge theory there is a non-zero net flow of magnetic fields across closed surfaces, $\oiint F \neq 0$, yet at the same time the magnetic charge density completely vanished $P = DF = DDG = 0$ just like in abelian gauge theory. To reconcile this, we determined that the magnetic charge density in non-abelian gauge theory is not the elementary $P = DF = DDG = 0$, but rather is a composite *faux* magnetic charge density $P' = -id\left[G,G\right] = -idGG$ constructed from gauge fields, and particularly, that the net flux of magnetic field is given by $\oiint F = -i\oiint\left[G,G\right] \neq 0$ in (3.3).

Ever since then, we have known that non-abelian gauge theory gives rise to a non-zero $\oiint F \neq 0$, but beyond a few vague hints pointing in the possible direction of baryons and confinement, it has not been known what the physics of this $\oiint F \neq 0$ might be. Now, we see in (10.5) that $\oiint \mathrm{Tr}\Sigma F\left(0\right) = -i\oiint \mathrm{Tr}\Sigma\left[G,G\right]\left(0\right) \sim \overline{R}R + \overline{G}G + \overline{B}B$. In other words, the composite faux magnetic fields which net flow across closed surfaces in non-abelian gauge theory are simply colorless mesons with the symmetric $\overline{R}R + \overline{G}G + \overline{B}B$ wavefunction. Colorless $\overline{R}R + \overline{G}G + \overline{B}B$ mesons – which, once flavored, include such things as the pions that mediate nuclear interactions – are simply the $\oiint F \neq 0$ faux magnetic monopole fields of Yang-Mills gauge theory. That means that these $\mathrm{Tr}\Sigma iF_{\mathrm{eff}\,\mu\nu} = \mathrm{Tr}\Sigma\left[G_\mu,G_\nu\right]$ objects in (10.4) – which are the only objects which flow in and out of the monopoles – *must be the mediators of interactions between the monopoles*. So if those monopoles are baryons as suggested by their $R\left[G,B\right] + G\left[B,R\right] + B\left[R,G\right]$ wavefunctions, and if these baryons can be turned into protons and neutrons as well shall show how to do momentarily, then these $\mathrm{Tr}\Sigma iF_{\mathrm{eff}\,\mu\nu} = \mathrm{Tr}\Sigma\left[G_\mu,G_\nu\right]$ fields are also the mediators of the nuclear interaction. And this also means that we should look to $\mathrm{Tr}\Sigma iF_{\mathrm{eff}\,\mu\nu} = \mathrm{Tr}\Sigma\left[G_\mu,G_\nu\right]$ when studying anything that might pass in and out of a proton or neutron through a closed $\oiint$ surface including energies released during nuclear fusion and fission which of course are intimately related to nuclear binding energies.

Related to this, to ensure Exclusion for the fermions in (9.8), we were forced to introduce a rank-3 gauge group which we assumed to be $SU(3)_C$. As pointed out after (9.16), after shifting the degrees of freedom using a Goldstone-like mechanism, this yielded eight associated gauge fields, which are bi-colored and massless, just like the strong interaction gluons. As had been earlier shown at (3.5), the abelian properties of the differential geometry via $dd = 0$ which is responsible in electrodynamics for the absence of magnetic monopoles entirely, prevents individual gauge fields – now these eight bi-colored massless gauge fields – from net flowing across any closed surface of the faux magnetic monopole P' because of $\oiint dG = \iiint R^{\tau}{}_{\nu\sigma\mu} G_{\tau} dx^{\sigma} dx^{\mu} dx^{\nu} = \mathbf{0}$. So in this way, these eight bi-colored massless gauge fields appeared to be *confined*. What we now see more explicitly and deeply in (10.5) is that the only thing which *does* net flow across these closed surfaces, are mesons which possess a color wavefunction $\overline{R}R + \overline{G}G + \overline{B}B$. And finally we saw at the start of this section that the faux magnetic monopoles themselves possess the totally-antisymmetric color wavefunction of a baryon, namely, $R[G,B] + G[B,R] + B[R,G]$. While one may think of this as color "confinement," what it really says is that is that the non-abelian faux magnetic monopoles P' and the mesons $[G,G]$ which net flow across closed surfaces of these monopoles, respectively, are antisymmetrically and symmetrically *color neutral*, and that nothing is permitted to net-flow across a closed monopole surface *unless* it has a $\overline{R}R + \overline{G}G + \overline{B}B$ neutral color configuration. So individual gauge fields, because they are bi-colored and not color neutral, are confined.

With all of this, we see multiple symmetries which are highly reminiscent of hadron physics: We are forced to introduce three fermion eigenstates which can be arbitrarily named as three "colors" just like the quark fields which transform non-trivially under SU(3) in the chromodynamic theory of strong interactions. What is arbitrary are the names; *what is not arbitrary is that we require three such names.* This simultaneously produces eight bi-colored gauge fields, also transforming non-trivially under SU(3), just as is the case for the strong interaction gluons, and so *derives* the chromodynamic requirement for a theory with three colors of fermion and eight bi-colors of gluon, and shows why baryons contain three quarks. These gluons after using the Goldstone-like mechanism in (9.16) must become massless just like the strong interaction gluons. The faux magnetic monopoles (9.21) have the antisymmetric, color-neutral symmetry of a baryon, and so are SU(3)-invariant. No gauge fields are allowed to net flow across any closed surface of this monopole, which means that the gauge fields are "confined" within the closed monopole surface, just like individual gluons. Yet there *is* a net flux of a non-abelian magnetic field across the closed monopole surfaces, as we found all the way back in section 3. Now, we see that these net-flowing magnetic fields have the symmetric, color-neutral symmetry of a meson, which means that they too are SU(3)-invariant, and that interactions between the faux monopoles will take place via colorless meson exchange, exactly as occurs in strong hadronic interactions between baryons.

Or, as Jaffe and Witten make clear at page 3 of [6], "quark confinement" is evidenced when:

> "even though the theory is described in terms of elementary fields, such as the quark fields, that transform non-trivially under SU(3), the physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant."

This is exactly what transpires if one regards the composite faux magnetic monopole of (9.21) as a zero-perturbation, ground state baryon density! Given all of these symmetries, from here we shall regard the monopole $\mathrm{Tr}\Sigma P'\left((0)\right)_0$ as a ground state baryon. And this means that (9.3), and specifically $\mathrm{Tr}\Sigma P'\left((0)\right)_\infty$, which contains $\pi_\infty = \left(\pi_0{}^{-1} + \pi_{\infty-1} J_\tau k^\tau + \pi_{\infty-1} J_\tau \pi_{\infty-1} J^\tau\right)^{-1}$ which can be expanded using (8.18) to reveal an exceptionally-non-linear system with perturbations up to infinite order in current density J and gauge field momentum k, is the *physical baryon* with all of its non-linear quark and gluon field behaviors.

Proceeding forward, we now expand the differential forms relationship for the faux magnetic charge density $P' = -id\left[G,G\right] \quad \left(=-idGG\right)$ uncovered after (3.3) into tensor form, expand $G_\mu = \lambda^i G^i{}_\mu$, and then, having extracted the group generators, finally apply the SU(3) group relation $\left[\lambda^i,\lambda^j\right] = if^{ijk}\lambda^k$. This yields:

$$
\begin{aligned}
P'_{\sigma\mu\nu} &= -i\left(\partial_\sigma\left[G_\mu,G_\nu\right] + \partial_\mu\left[G_\nu,G_\sigma\right] + \partial_\nu\left[G_\sigma,G_\mu\right]\right) \\
&= -i\left[\lambda^i,\lambda^j\right]\left(\partial_\sigma\left(G^i{}_\mu G^j{}_\nu\right) + \partial_\mu\left(G^i{}_\nu, G^j{}_\sigma\right) + \partial_\nu\left(G^i{}_\sigma, G^j{}_\mu\right)\right). \\
&= f^{ijk}\lambda^k\left(\partial_\sigma\left(G^i{}_\mu G^j{}_\nu\right) + \partial_\mu\left(G^i{}_\nu, G^j{}_\sigma\right) + \partial_\nu\left(G^i{}_\sigma, G^j{}_\mu\right)\right)
\end{aligned}
\tag{10.6}
$$

Let us now *assume* as we have since after (9.9) that our gauge group is the simple *sub*group SU(3) with the eight traceless generators λ^k, $k=1...8$ often referred to as the Gell-Mann matrices. If we now take the trace of the above, given that the eight λ^k of the *sub*group SU(3) are all traceless, $\mathrm{Tr}\lambda^k = 0$, (10.6) tells us that $\mathrm{Tr}P'_{\sigma\mu\nu} = 0$.

But (9.21) has a non-zero trace, and so it is worthwhile understanding how it is that even when we assume an SU(3) subgroup with $\mathrm{Tr}\lambda^k = 0$, we can still end up with a non-zero trace equation (9.21). The key is to closely examine (9.7), which is why we chose to display the intermediate terms even though we could have gone directly from (9.6) to the bottom line (9.7) using $J^\mu = \overline{\Psi}\gamma^\mu\Psi$ without showing generators or internal symmetry indexes. The key is that (9.6) contains commutators $\left[J^\mu, J^\nu\right]$, and so contains a very specific type of second-order expression for the currents J^ν. Although the generators are traceless, when any generator is squared and then traced, the result in the customary normalization is the non-zero $\mathrm{Tr}\left(\lambda^i\right)^2 = \frac{1}{2}$. In the intermediate terms (9.7), we see multiple sums $\lambda^i\lambda^i$ of a generator with itself. When all of the anti-symmetries in these intermediate terms are accounted for, the result is the bottom line of (9.7) which, by the time it is worked into (9.21), reflects in a deeper way of the general result that $\mathrm{Tr}\left(\lambda^i\right)^2 = \frac{1}{2}$ is not zero.

Nonetheless, (10.6) appears to contradict this non-zero trace result obtained in (9.21) wherein $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left(0\right) \neq 0$. This is another puzzle. But think about this more closely: In (9.9) we were compelled to introduce a rank-3 gauge group to enforce exclusion for each of the fermion wavefunctions in (9.8). But all we really knew is that we needed three mutually-exclusive eigenstates and therefore required a rank-3 gauge group. Although we could have just as readily chosen SU(3)×U(1), we *assumed* that the gauge group could be SU(3) unless and until contradicted. But now this assumption *is* contradicted. Specifically, based on the development up to (9.8), the choice of a gauge group appeared to be non-unique. Any rank-3 group would do. But by the time we reached (9.21), it became clear that we had a $\mathrm{Tr}P'_{\sigma\mu\nu} \neq 0$, i.e., that $P'_{\sigma\mu\nu}$ must have a non-vanishing trace. If one tries to write (9.21) in the same way as (10.6) to extract out an overall $f^{ijk}\lambda^k$, it cannot be done, other than by backtracking to (9.7). The development from (9.7) (where this still could be done) to (9.21) removed the ability to do so, and in particular, that started to happen once we used (9.12) in (9.13) and summed spins to remove two wavefunctions using the fermion spin sum.

Now, (10.6) informs us that if the gauge group is SU(3) then the trace will vanish. So now, what appeared at (9.9) to be a non-unique choice of SU(3) or SU(3)×U(1) is forced by (9.21) in view of (10.6) to be a *unique* choice of SU(3)×U(1), with λ^0 used to denote the new U(1) generator, which now also adds one more degree of freedom to the (9.21) system. Of course, we will now need to determine what this additional U(1) generator represents, and as we shall see, it represents the baryon number $B=1/3$ for each of the three colored fermions appearing in (9.21) and may be used to more formally turn the faux magnetic monopole density (10.6) into a baryon density. As we shall also see, while the gauge group SU(3) by itself is simply the usual color group $\mathrm{SU(3)_C}$ of strong interaction chromodynamic theory, once this group gets crossed with U(1) it becomes a "modified" color group which mixes color and *flavor* because the introduction of baryon number also facilitates the introduction of the flavor-distinguishing electric charge generator Q. But before we discuss this, there is a more general point that must be made, and this has to do with topological stability.

Cheng and Li point out at 472-473 of [17] that "topological considerations lead to the general result that stable monopole solutions occur for any gauge theories in which a *simple* gauge group G is broken down to a smaller group $H = h \times \mathrm{U(1)}$ containing an explicit U(1) factor." Further, "the stable grand unified monopole . . . is expected to have both the 'ordinary' and the colour magnetic charges." So, while SU(3) alone is incapable of supporting a topologically-stable colored magnetic monopole, the group SU(3)×U(1) – when understood to be the residual group following symmetry breaking of a larger simple grand unified gauge group $G \supset \mathrm{SU(3)\times U(1)}$ – *will support topologically stable configurations*. This is an essential requirement if the faux monopole (10.6) can ever be regarded as a physically-stable entity like a baryon, and especially a distinctively-stable proton, and a neutron which is comparatively stable when free, and very stable when part of many lighter atomic nuclei.

Weinberg makes a similar point to Cheng and Li in his definitive treatise [18] at 442:

> "The Georgi-Glashow model was ruled out as a theory of weak and electromagnetic interactions by the discovery of neutral currents, but magnetic monopoles are expected to occur in other theories, where a simply connected gauge group G is spontaneously broken not to U(1), but to some subgroup

> *H'*×*U*(1), where *H'* is simply connected. . . . There are no monopoles produced in the spontaneous breaking of the gauge group *SU*(2)×*U*(1) of the standard electroweak theory, which is not simply connected. . . . But we do find monopoles when the simply connected gauge group *G* of theories of unified strong and electroweak interactions, such as *SU*(4)×*SU*(4) or *SU*(5) or *Spin*(10), is spontaneously broken to the gauge group *SU*(3)×*SU*(2)×*U*(1) of the standard model. . . ."

Consequently, not only does (9.8) force us to uniquely select a rank-3 gauge group to enforce Exclusion on the faux magnetic monopole density of (9.8), but the non-vanishing trace of (9.21) forces us into the *specific, unique selection* of SU(3)×U(1) over SU(3). This then ensures that these faux monopoles will be topologically stable so long as we arrive at this product group following the spontaneous symmetry breaking of a larger simple gauge group $G = SU(N \ge 4) \supset \mathrm{SU(3)\times U(1)}$, as yet undetermined. Topologically speaking, referring again to Weinberg's [18] at 442, the homotopy groups associated with this symmetry breaking would be:

$$\pi_2\left(G / SU(3)\times U(1)\right) = \pi_1\left(SU(3)\times U(1)\right) = \pi_1\left(SU(3)\right)\times\pi_1\left(U(1)\right) = \pi_1\left(U(1)\right) = Z \,. \tag{10.7}$$

So there are really two questions raised by the non-vanishing trace in (9.21). First, as already stated, what is the physical meaning of the new U(1) generator? Second, what is the larger group $G = SU(N \ge 4) \supset \mathrm{SU(3)\times U(1)}$ from which we arrive at SU(3)×U(1) following symmetry breaking so as to achieve topological stability? There is also a third question, not yet apparent, but linked to the first question, which is this: what is the meaning of the $SU(3)$ group which is multiplied by the new U(1) gauge group as part of $SU(3)\times U(1)$, and how does this relate to the usual color group $\mathrm{SU(3)_C}$?

For the new U(1) group which provides topological stability, the generator λ^0 must be a constant multiple of the 3x3 identity (unit) matrix $I_{3\times3}$. If we normalize this to $\mathrm{Tr}\left(\lambda^0\right)^2 = \frac{1}{2}$ just like all the other generators, then we must have $\lambda^0 = \frac{1}{\sqrt{6}} I_{3\times3}$. Taken together with the two remaining diagonalized generators of SU(3) normalized to $\mathrm{Tr}\left(\lambda^i\right)^2 = \frac{1}{2}$, we have:

$$\lambda^0 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad \lambda^3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10.8}$$

But that is only the mathematics: now we need a *physical* interpretation for λ^0. Because each of the three fermion eigenstates in (9.9) will have identical λ^0 eigenvalues, because the monopole in (9.21) exhibits many of symmetries of a baryon and the fermions exhibit many of the symmetries of quarks, it would appear fruitful to assign the U(1) generator to baryon number *B* according to:

$$B \equiv 2\tfrac{1}{\sqrt{6}}\lambda^0 = \tfrac{1}{3}I_{3\times 3} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{10.9}$$

This is our first explicit introduction of *flavor* into the color eigenstates that were introduced at (9.9). Following (10.9), the monopole (9.21) will now have $B=1$ and each of the R, G, B fermions will now have $B=\frac{1}{3}$, which brings these even a step closer to being identifiable with baryons and quarks.

Next, if these monopoles (9.21) are to be baryons and the fermions are to be quarks, let us see if there is some way to identify the *electric charge Q* of these baryons, and specifically to produce a proton with $Q=+1$ which has a duu configuration of quark *flavors*, and a neutron with $Q=0$ which has a udd configuration of quark flavors, wherein the up (u) quark has $Q=+\frac{2}{3}$ and the down (d) quark has $Q=-\frac{1}{3}$.

For the proton, we may form the combination:

$$Q_P \equiv B - \frac{2}{\sqrt{3}}\lambda^8 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}, \tag{10.10}$$

Following (10.9), each of the R, G, B colored fermions in (9.9) has a flavored baryon number $B=\frac{1}{3}$. Now, with (10.10), the red *color* of fermion is assigned $Q=-\frac{1}{3}$ and so is a down *flavor* of fermion in addition to its red color assignment, the green and blue *colors* of quark are assigned $Q=+\frac{2}{3}$ and so are up *flavors* of fermion in addition to their green and blue color assignments. So the SU(3)×U(1) quark triplet is now $\left(d_R, u_G, \mu_B\right)$. Further, the entire faux monopole $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left(\left(0\right)\right)_0$ of (9.21) which comprises all of these fermions has a baryon number $B=1$ and an electric charge $Q=+1$ and so is a proton-flavored baryon with the color-neutral wavefunction $R\left[G,B\right]+G\left[B,R\right]+B\left[R,G\right]$. To use a parlance familiar from electroweak theory, we see in (10.10) that the electric charge generator for the proton and for the quarks within the proton *sit across* baryon number *B* and the λ^8 color generator, that is, they sit across SU(3)×U(1) in a *non-compact* manner. In similar fashion, in electroweak theory a $\mathrm{U(1)_Y}$ generator is crossed with the three $\mathrm{SU(2)_W}$ isospin generators I^i, $i=1,2,3$ to form $\mathrm{SU(2)_W}\times\mathrm{U(1)_Y}$ with the (left-chiral) quark doublets having the $\mathrm{U(1)_Y}$ 2x2 weak hypercharge matrix generator $Y=\frac{1}{3}I_{2x2}$, the (left-chiral) lepton doublets having the 2x2 weak hypercharge matrix generator $Y=-1I_{2x2}$, and a *non-compact* embedding of the electromagnetic group with charge generator $Q=Y/2+I^3$ *sitting across* $\mathrm{SU(2)_W}\times\mathrm{U(1)_Y}$.

For the neutron it is even simpler. We simply make the *compact* assignment:

$$Q_N \equiv \frac{2}{\sqrt{3}}\lambda^8 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{10.11}$$

Here, all of the fermions still have baryon number $B=\frac{1}{3}$. But now the red fermion is assigned $Q=+\frac{2}{3}$ thus is an up flavored-fermion, the green and blue fermions are assigned $Q=-\frac{1}{3}$ and so are down flavored. So the quark triplet is now (u_R, d_G, d_B). The overall faux monopole of (9.21) now has baryon number $B=1$ and electric charge $Q=0$ and so is a neutron-flavored baryon. So the electric charge generator for the neutron and its quarks is compactly-embedded in λ^8 which now serves the dual role of one of two SU(3)$_C$ generators *and* the electric charge generator.

Of course, the fact that we must employ a different charge assignment (10.10) for the proton than (10.11) for the neutron is symptomatic that there is a larger yet-to-be-found gauge group which encompasses the SU(3)×U(1) group developed in (10.8) through (10.11). That is $Q_P = B - \frac{2}{\sqrt{3}}\lambda^8$ and $Q_N = \frac{2}{\sqrt{3}}\lambda^8$ is not invariant whereby one relationship, not two, defines the relationship between the electric charge and the group generators. This disconnection between the proton and neutron electric charges is analogous to how in electroweak theory, the $Y_q = \frac{1}{3}$ for the quark (q) doublets is disconnected from the $Y_l = -1$ lepton (l) doubles which there too, signifies the need for a larger unifying group. So the question is now raised: what is the nature of the gauge group that provides a unified basis for the proton and neutron electric charges *Q*, and can this same group *also* provide the basis for unifying the separate *Y* charges as between quarks and leptons while also dealing with chiral symmetry (breaking) issues?

While we shall not explore this here, the author has studied these exact questions in [19] and shown how a simple SU(8) group with the fundamental fermion multiplet $(\nu, u_R, d_G, d_B, e, d_R, u_G, u_B)$ provides a complete unification which breaks down at low energies to the phenomenological SU(3)$_C$×SU(2)$_W$×U(1)$_Y$ with protons and neutrons, and at the same time – because two of the diagonalized SU(8) generators themselves become "fractured" apart from the other five diagonalized generators during symmetry breaking – leads to an explanation of why the known fermions appear to exist in exactly three generations, which answers Isador Rabi's famous quip about the muon "who ordered this?" That is because these two "fractured" generators provide the precise freedom needed to accommodate three horizontal generational eigenstates.

But what we now know from the development within this paper and specifically (10.10) and (10.11) is that the SU(3)$_C$ group which we introduced at (9.9) to enforce Exclusion actually becomes modified into a *hybrid color and flavor group* in view of the requirement to use SU(3)×U(1) because of the non-vanishing trace in (9.21). We shall thus refer to this as a "flavor-enhanced color group" which we denote generally by SU(3)$_{C'}$. When we use this group to represent a proton (P) quark triplet (d_R, u_G, μ_B) with the charge assignments (10.10) we shall further denote this by SU(3)$_{PC'}$, while when we use this to represent a neutron (N) quark triplet

(u_R, d_G, d_B) with the charge assignments (10.11) we shall denote this by $SU(3)_{NC'}$. Finally, in all cases, the U(1) factor is associated with baryon number *B*, so we shall denote this as $U(1)_B$. So to summarize, once the U(1) factor is in place, the group developed thus far is $SU(3)_{C'}\times U(1)_B$. For protons it is specialized via (10.10) to $SU(3)_{PC'}\times U(1)_B$. For neutrons it is specialized via (10.11) to $SU(3)_{NC'}\times U(1)_B$.

Next, keeping in mind (10.7), it also becomes important to find a larger simple gauge group $G = SU(N \geq 4) \supset \mathrm{SU}(3)_{C'}\times \mathrm{U}(1)_B$ which breaks down spontaneously to $SU(3)_{C'}\times U(1)_B$. As the author details in section 7 of [15], there are two disconnected $G = SU(4)$ groups, but we are able to use $B - L \equiv -\sqrt{\tfrac{8}{3}}\lambda^{15}$ as the generator of baryon minus lepton number for both. This follows Volovok from [20] Section 12.2.2 who also uses the λ^{15} of SU(4) for a $B-L$ generator, but in the context of a preon model. The first group, denoted $SU(4)_P$, places the proton's quarks and the electron into a (e, d_R, u_G, u_B) quadruplet in the fundamental representation. The second group, denoted $SU(4)_N$, places the neutron's quarks and the neutrino into a (ν, u_R, d_G, d_B) quadruplet in the fundamental representation. Then, each of these disconnected proton and neutron groups gets broken at GUT energies via $G = SU(4)_{B-L} \rightarrow \mathrm{SU}(3)_{C'}\times \mathrm{U}(1)_B$ to produce the stable magnetic monopole baryons via:

$$\pi_2\left(SU(4)_{B-L} / SU(3)_{C'}\times U(1)_B\right) = \pi_1\left(SU(3)_{C'}\times U(1)_B\right) = \pi_1\left(SU(3)_{C'}\right)\times \pi_1\left(U(1)_B\right) = \pi_1\left(U(1)_B\right) = Z\ . \quad (10.7)$$

Then, as the author details throughout [19], the disconnected $SU(4)_N$ and $SU(4)_P$ groups become unified together in the $(\nu, u_R, d_G, d_B, e, d_R, u_G, u_B)$ of SU(8) mentioned moments ago, such that two of the seven generators (λ^{48} and λ^{35}) become fractured from the remaining generators between the Planck and the GUT energy scales to provide the "horizontal" degrees of freedom needed to accommodate replication of the fermions into three generations, and there is also just enough freedom provided to also support chiral symmetry breaking. Additionally, all of the observed features of left-chiral Cabbibbo / CKM mixing naturally emerge. The overall sequence of symmetry breaking is:

$$SU(8) \rightarrow SU(6)_B \times SU(2)_L \rightarrow SU(3)_C \times SU(2)_W \times U(1)_{Y_L = B-L} \rightarrow SU(3)_C \times U(1)_{em}\ . \quad (10.12)$$

Simultaneously with and as part of the $SU(8) \rightarrow SU(6)_B \times SU(2)_L$ symmetry breaking, the two isospin-differing $SU(4)_{B-L} \rightarrow \mathrm{SU}(3)_{C'}\times \mathrm{U}(1)_B$ symmetry breaks also take place to form the topologically-stable proton and neutron. There is also an earlier breaking of $SU(8) \rightarrow SU(7)\times U(1)$ at or near Planck energies which separates the neutrino from all the other fermions right at the very start and causes the neutrino to behave very differently from all the other fermions as it clearly does at observable energies. The symmetry breaking sequences found in [19] are then utilized in [21] to explain the observed proton and neutron masses themselves in relation to the current up and down quark masses and the CKM mixing matrices based on [16], *within all experimental errors*.

Next, let us return to (9.4) where we set the perturbation to $V=0$ in (9.2). Because everything that has been developed since (9.2), most notably the $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left((0)\right)_0$ monopole / baryon of (9.21) was developed for $V=0$, the question may be asked whether all of these results carry through when we no longer set $V=0$ but allow all of the perturbations to occur. Section 8 answers this question. What we learn in section 8 is that including perturbations really means recursing $G_\mu=\left(k_\tau k^\tau-m^2+i\varepsilon+G_\tau k^\tau+G_\tau G^\tau\right)^{-1}J_\mu$ as many times as one chooses, then cutting off the recursion by setting $V=-G_\tau k^\tau-G_\tau G^\tau=0$ at some chosen recursive order. Of course, recursing to some order *n* and then setting *V*=0 as in (8.17) and (8.18) to arrive at a $\left((0)\right)_n$ expression is a calculation technique. But it is to be expected that nature does not cut off the recursion at all, but rather, recurses to infinity before setting $V=0$, so that $G_\mu=\pi_\infty J_\mu$ as in (8.20). So if the monopole $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left((0)\right)_0$ of (9.21) is the ground state of the baryon, it will be the infinite recursion of (8.20), not some arbitrarily truncated recursion, which will drive what nature herself does in physical reality. This means that (9.3) in the form $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left((0)\right)_\infty$, is really the equation for the *physical* baryon, with a teeming non-linear mix of quarks and gauge fields in a "sea" perturbating through all finite orders up to infinite order, which is exactly what one observes in the complex composite system that is a proton or a neutron or any other baryon.

Finally, although (9.21), if it represents a baryon, only does so in the zero-perturbation, no-recursion limit, it is important to ask whether there is anything about this limit that is observed in nature. Put differently, while cutting off the perturbations at the zeroth recursive order may see arbitrary, it is the only order beside infinite order that would seem to have some distinctive claim to not being arbitrary. And so we raise the question whether there are any phenomena observed in nuclear or particle physics which manifest the linear, non-perturbative behavior of the $\mathrm{Tr}\Sigma P'^{\sigma\mu\nu}\left((0)\right)_0$ baryon (9.21)? To use an analogy, although gravitation is a highly non-linear theory, we do observe certain aspects of the linear behavior of gravitation theory in the real world, namely, whenever we observe what was first discovered by Keppler and Newton. So while we would most certainly need to describe the complete proton and neutron and other baryons without removing the perturbations from (9.2) a.k.a. (9.3), we should also look to see if certain aspects of nuclear behavior that might be very-definitively described by the "linear approximation" (9.21).

In this regard, $F_{\mathrm{eff}\,\mu\nu}\left((0)\right)_0$ in (10.4) is very important for pursuing experimental validation, because it does describe what "effectively" net flows in and out of the closed monopole surfaces in the ground state linear theory. Specifically, it is well-known that one can calculate electrodynamic energies from the pure gauge field $\mathfrak{L}_{\mathrm{gauge}}=-\frac{1}{4}F_{\sigma\tau}F^{\sigma\tau}$ by using this in $E=-\iiint\mathfrak{L}_{\mathrm{gauge}}d^3x$. So one should do a similar exercise using what in non-abelian theory becomes the Lagrangian density $\mathfrak{L}_{\mathrm{gauge}}=-\frac{1}{2}\mathrm{Tr}\left(F_{\sigma\tau}F^{\sigma\tau}\right)$, using $F_{\mathrm{eff}\,\mu\nu}\left((0)\right)_0$. If we compare (10.4) which is a trace equation to (9.21) which is another trace equation from which it was derived, then by backtracking to (9.20), we see that (we have now removed the Σ spin sum designation, which now is taken to be implied):

$$F_{\text{eff}\,\mu\nu}\left(\left(0\right)\right)_0 = -i\begin{pmatrix} \overline{\psi_R}\gamma_{[\mu}\left(p_R - m_R\right)^{-1}\gamma_{\nu]}\psi_R & 0 & 0 \\ 0 & \overline{\psi_G}\gamma_{[\mu}\left(p_G - m_G\right)^{-1}\gamma_{\nu]}\psi_G & 0 \\ 0 & 0 & \overline{\psi_B}\gamma_{[\mu}\left(p_B - m_B\right)^{-1}\gamma_{\nu]}\psi_B \end{pmatrix}. \quad (10.13)$$

This is now a 3x3 matrix expression with all diagonal elements. From this, there are two trace expressions that can be formed. One is $\mathrm{Tr}\left(F_{\sigma\tau}F^{\sigma\tau}\right)$ which is what is usually found in the Yang-Mills Lagrangian density. The other is $\mathrm{Tr}F_{\sigma\tau}\mathrm{Tr}F^{\sigma\tau}$.

It turns out as the author has detailed in sections 11 and 12 of [15], and greatly expanded upon throughout [16], that the expression (10.13) when used in $E = -\iiint \mathfrak{L}_{\text{gauge}} d^3x$ with a combination of $\mathrm{Tr}\left(F_{\sigma\tau}F^{\sigma\tau}\right)$ and $\mathrm{Tr}F_{\sigma\tau}\mathrm{Tr}F^{\sigma\tau}$ inner and outer products, can be used to retrodict *nuclear binding energies*, including the heretofore unexplained binding energies of the lightest nuclides ^{2}H, ^{3}H, ^{3}He and ^{4}He, as well as the ^{56}Fe binding energy, with parts per 10^5 or even 10^6 AMU precision, and the neutron minus proton mass difference to *under one part per million AMU*. Note that in general, the trace of a product of two square matrices is *not* the product of traces. The only circumstance in which "trace of a product" equals "product of traces" is when one forms a tensor outer product using $\mathrm{Tr}\left(A \otimes B\right) = \mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right)$, and as shown in [16] the observed binding energies contain both inner and outer products. This line of development in sections 11 and 12 of [15] and throughout [16] also explains why the per-nucleon binding energy seems to be limited for any nucleus to a maximum of about 8.75 MeV for ^{56}Fe, and yields a dynamical, energy-based understanding of confinement.

While all of the formal understandings of the color symmetries of baryons and mesons and quarks are important, direct experimental validation is even more important. It is the experimental concurrences that can be confirmed starting with (10.13) to perform various energy calculations $E = -\iiint \mathfrak{L}_{\text{gauge}} d^3x$ with $\mathrm{Tr}\left(F_{\sigma\tau}F^{\sigma\tau}\right)$ and $\mathrm{Tr}F_{\sigma\tau}\mathrm{Tr}F^{\sigma\tau}$, that leads to the direct phenomenological confirmation that the faux magnetic monopoles of non-abelian gauge theory really are baryons including protons and neutrons.

11. Quantum Yang-Mills Theory: Exact Analytical Path Integration

Finally, let us make use of the recursion developed in section 8, and particularly the substitution $G_\mu \to \left(\pi_0^{-1} + \pi_{\infty-1}J_\tau k^\tau + \pi_{\infty-1}J_\tau \pi_{\infty-1}J^\tau\right)^{-1}J_\mu$ from (8.20) in lieu of the usual $G_\mu \to \delta / \delta J^\mu$, to perform an *exact analytical* deduction of the quantum path integral associated with the classical field equation $-J^\nu = \left(g^{\nu\sigma}D_\tau D^\tau - D^\sigma D^\nu\right)G_\sigma$ of (5.15) in order to "prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$," see page 6 of Jaffe and Witten's [6].

In abelian gauge theory, the classical electric charge field equation is of course $*J = d*dG$ which is an abelian subset equation embedded in (1.12). When fully expanded for a massive boson this becomes the abelian $-J^\nu = \left(g^{\nu\sigma}\left(\partial_\tau\partial^\tau + m^2\right) - \partial^\sigma\partial^\nu\right)G_\sigma$ of (5.15). The related action after integration-by-parts is thus $S(G) = \tfrac{1}{2}G_\mu\left(g^{\mu\nu}\left(\partial_\sigma\partial^\sigma + m^2\right) - \partial^\mu\partial^\nu\right)G_\nu + J^\mu G_\mu$, and this is what is used in the path integral $Z = \int DG \exp i\int d^4x S(G) \equiv \exp iW(J)$ to deduce the quantum amplitude $W(J) = \tfrac{1}{2}\int\left(d^4k/(2\pi)^4\right)J_\sigma\left(k_\sigma k^\sigma - m^2 + i\varepsilon\right)^{-1}J^\sigma$ with $+i\varepsilon$ using the contextual reduction that also occurs from the continuity relation $k_\nu J^\nu = 0$ as reviewed at length in section 6 and 7. If we use the terminal condition $\pi_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1}$ of the (8.20) recursion, then this simplifies to $W(J) = \tfrac{1}{2}\int\left(d^4k/(2\pi)^4\right)J_\sigma\pi_0 J^\sigma$.

In non-abelian gauge theory the classical field equation is the entirety of (1.12), that is $*J = D*DG$ which as shown expands to $-J^\nu = \left(g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu\right)G_\sigma$ derived in (5.15). Without going through a detailed exposition of how to derive the associated Lagrangian and conduct the integration-by-parts to obtain the action, it will be appreciated that as the result of this exercise the non-abelian action will found to be:

$$\begin{aligned} S(G) &= G_\mu\left(g^{\mu\nu}\left(D_\tau D^\tau + m^2\right) - D^\mu D^\nu\right)G_\nu + 2J^\tau G_\tau \\ &= G_\mu\left(g^{\mu\nu}\left(\left(\partial_\tau\partial^\tau - iG_\tau\partial^\tau - G_\tau G^\tau\right) + m^2\right) - \left(\partial^\mu\partial^\nu - iG^\mu\partial^\nu - 2G^\mu G^\nu + G^\mu G^\sigma\right)\right)G_\nu + 2J^\tau G_\tau \end{aligned}, \tag{11.1}$$

where we have also included (5.16) and (5.17).

When we now take the next step of using this action in $Z = \int DG \exp i\int d^4x S(G)$, there are now two new issues that come into play that are not present in the abelian gauge theory. The first is that the non-abelian gauge transformation $G^\nu \to G'^\nu = G^\nu + \partial^\nu\theta - i\left[G^\nu, \theta\right]$ gives rise to ghost fields due to the introduction of the additional term $-i\left[G^\nu, \theta\right]$ into the integration measure DG in order to ensure that $Z \to Z' = Z$ remains invariant under this gauge transformation, and so we need to employ $DGDcDc^\dagger$ not just DG as the integration measure. But the second issue is that even before we get to worrying about ghost fields, it is simply not known, as a mathematical matter, how to use an expression like (11.1) in a path integral to calculate:

$$\begin{aligned} Z &= \int DG \exp i\int d^4x\left[G_\mu\left(g^{\mu\nu}\left(D_\tau D^\tau + m^2\right) - D^\mu D^\nu\right)G_\nu + 2J^\tau G_\tau\right] \\ &= \int DG \exp i\int d^4x\left[G_\mu\begin{pmatrix} g^{\mu\nu}\left(\left(\partial_\tau\partial^\tau - iG_\tau\partial^\tau - G_\tau G^\tau\right) + m^2\right) \\ -\left(\partial^\mu\partial^\nu - iG^\mu\partial^\nu - 2G^\mu G^\nu + G^\mu G^\sigma\right) \end{pmatrix}G_\nu + 2J^\tau G_\tau\right] \end{aligned}. \tag{11.2}$$

This is because, as will be apparent from studying the lower expression, this is a fourth-order polynomial in G, but known mathematical techniques for calculating integrals of this form use the second order $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right) = \left(-2\pi / A\right)^{.5} \exp\left(J^2 / 2A\right)$. Why? Put plainly and simply, it is known how to calculate $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right)$, but *not* how to calculate the higher order $\int dx \exp\left(Bx^4 + Cx^3 - \frac{1}{2}Ax^2 - Jx\right)$. Normally, of course, the approach is to turn every gauge field inside the configuration space operator $g^{\mu\nu}\left(D_\tau D^\tau + m^2\right) - D^\mu D^\nu$ into a current term $G_\mu \to \delta / \delta J^\mu$ and then use (8.25) to apply $\exp\left(-V\left(\delta / \delta J\right)\right)$ to $\exp\left(\frac{1}{2} J \cdot K^{-1} \cdot J\right)$ the latter of which is obtained in the usual way from $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right) = \left(-2\pi / A\right)^{.5} \exp\left(J^2 / 2A\right)$.

But now the recursion developed in section 8 gives us a new *mathematical* approach. Now, we are able to use (8.20) to turn every occurrence of G inside $g^{\mu\nu}\left(D_\tau D^\tau + m^2\right) - D^\mu D^\nu$ into a function solely of $G\left(J,k\right)$ via $G_\mu = \pi_\infty J_\mu = \left(\pi_0^{-1} + \pi_{\infty-1} J_\tau k^\tau + \pi_{\infty-1} J_\tau \pi_{\infty-1} J^\tau\right)^{-1} J_\mu$ with the abelian terminal condition $\pi_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1}$. *None of these contain* G_μ *!* So, making this replacement in (11.2), we now have

$$
\begin{aligned}
Z &= \int DG \exp i \int d^4x \left[G_\mu \left(g^{\mu\nu}\left(D_\tau D^\tau + m^2\right) - D^\mu D^\nu \right) G_\nu + 2J^\tau G_\tau \right] \\
&= \int DG \exp i \int d^4x \left[G_\mu \begin{pmatrix} g^{\mu\nu}\left(\left(\partial_\tau \partial^\tau - iG_\tau \partial^\tau - G_\tau G^\tau\right) + m^2\right) \\ -\left(\partial^\mu \partial^\nu - iG^\mu \partial^\nu - 2G^\mu G^\nu + G^\mu G^\sigma\right) \end{pmatrix} G_\nu + 2J^\tau G_\tau \right] \\
&= \int DG \exp i \int d^4x \left[G_\mu \begin{pmatrix} g^{\mu\nu}\left(\left(\partial_\tau \partial^\tau - i\pi_\infty J_\tau \partial^\tau - \pi_\infty J_\tau \pi_\infty J^\tau\right) + m^2\right) \\ -\left(\partial^\mu \partial^\nu - i\pi_\infty J^\mu \partial^\nu - 2\pi_\infty J^\mu \pi_\infty J^\nu + \pi_\infty J^\mu \pi_\infty J^\sigma\right) \end{pmatrix} G_\nu + 2J^\tau G_\tau \right] . \\
&\overset{i\partial \to k}{\Rightarrow} \int DG \exp i \int d^4x \left[G_\mu \begin{pmatrix} -g^{\mu\nu}\left(\left(k_\tau k^\tau + \pi_\infty J_\tau k^\tau + \pi_\infty J_\tau \pi_\infty J^\tau\right) - m^2\right) \\ +k^\mu k^\nu + \pi_\infty J^\mu k^\nu + 2\pi_\infty J^\mu \pi_\infty J^\nu - \pi_\infty J^\mu \pi_\infty J^\sigma \end{pmatrix} G_\nu + 2J^\tau G_\tau \right]
\end{aligned}
\tag{11.3}
$$

Lo and behold, we have removed all the gauge fields from the configuration space operator except for $G_\mu\left(...^{\mu\nu}\right)G_\nu$ and $J^\tau G_\tau$. This leaves us with the usual quadratic form $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right) = \left(-2\pi / A\right)^{.5} \exp\left(J^2 / 2A\right)$. So we can integrate this analytically and exactly, so long as we know the inverse for $\left(...^{\mu\nu}\right) = g^{\mu\nu}\left(D_\tau D^\tau + m^2\right) - D^\mu D^\nu$ or any of its other variants in (11.3). But this, of course, was a central focus of what we studied in section 6 and 7. Particularly, for the field equation $-J^\nu = \left(g^{\nu\sigma}\left(D_\tau D^\tau + m^2\right) - D^\sigma D^\nu\right)G_\sigma$, as seen in (8.19), with the context afforded by the continuity relation $D_\sigma J^\sigma = 0$, the inverse solution is simply $G_\mu = \pi_\infty J_\mu$. So we recognize immediately that the exact analytical solution to (11.3) is:

$$
\begin{aligned}
Z &= \int DG \exp i \int d^4x \left[G_\mu \left(g^{\mu\nu} \left(D_\tau D^\tau + m^2 \right) - D^\mu D^\nu \right) G_\nu + 2J^\tau G_\tau \right] \\
&= \int DG \exp i \int d^4x \left[G_\mu \begin{pmatrix} -g^{\mu\nu} \left(\left(k_\tau k^\tau + \pi_\infty J_\tau k^\tau + \pi_\infty J_\tau \pi_\infty J^\tau \right) - m^2 \right) \\ + k^\mu k^\nu + \pi_\infty J^\mu k^\nu + 2\pi_\infty J^\mu \pi_\infty J^\nu - \pi_\infty J^\mu \pi_\infty J^\sigma \end{pmatrix} G_\nu + 2J^\tau G_\tau \right]. \qquad (11.4) \\
&= \exp iW(J) = i \int \left(d^4k / (2\pi)^4 \right) J_\sigma \pi_\infty J^\sigma
\end{aligned}
$$

This, again, is an *exact analytical solution*. Expressed directly in terms of the amplitude and using (8.18), this means that:

$$
\begin{cases} (2\pi)^4 W(J) = \int d^4k J_\sigma \pi_\infty J^\sigma = \int d^4k J_\sigma \left(\pi_0^{-1} + \pi_{\infty-1} J_\tau k^\tau + \pi_{\infty-1} J_\tau \pi_{\infty-1} J^\tau \right)^{-1} J^\sigma \\ \pi_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon \right)^{-1} \end{cases} . \qquad (11.5)
$$

If it is desired to see explicitly how this gives us the non-linear propagator and current and momentum terms that we expect to find in a Yang-Mills path integral, it suffices, just for illustration, to examine the amplitude $W(J)_2$ for a second-order recursion, using the terminal condition $\pi_0 = \left(k_\tau k^\tau - m^2 + i\varepsilon \right)^{-1}$. This is (cf. (8.5)):

$$
\begin{aligned}
(2\pi)^4 W(J)_2 &= \int d^4k J_\sigma \pi_2 J^\sigma = \int d^4k J_\sigma \left(\pi_0^{-1} + \pi_1 J_\tau k^\tau + \pi_1 J_\tau \pi_1 J^\tau \right)^{-1} J^\sigma \\
&= \int d^4k J_\sigma \begin{pmatrix} \pi_0^{-1} + \left(\pi_0^{-1} + \pi_0 J_\tau k^\tau + \pi_0 J_\tau \pi_0 J^\tau \right)^{-1} J_\tau k^\tau \\ + \left(\pi_0^{-1} + \pi_0 J_\tau k^\tau + \pi_0 J_\tau \pi_0 J^\tau \right)^{-1} J_\tau \left(\pi_0^{-1} + \pi_0 J_\tau k^\tau + \pi_0 J_\tau \pi_0 J^\tau \right)^{-1} J^\tau \end{pmatrix}^{-1} J^\sigma \\
&= \int d^4k J_\sigma \begin{pmatrix} k_\tau k^\tau - m^2 + i\varepsilon \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + \dfrac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \dfrac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon \right)^2} \right)^{-1} J_\tau k^\tau \\ + \left(k_\tau k^\tau - m^2 + i\varepsilon + \dfrac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \dfrac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon \right)^2} \right)^{-1} J_\tau \\ \times \left(k_\tau k^\tau - m^2 + i\varepsilon + \dfrac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \dfrac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon \right)^2} \right)^{-1} J^\tau \end{pmatrix}^{-1} J^\sigma
\end{aligned} . \qquad (11.6)
$$

With this being only the second-order recursion, it will be appreciated how this will expand rapidly in a highly-nonlinear way to include all orders of J, k, m and $+i\varepsilon$, right through infinity. For doing practical calculations, including those with computers, one can use expressions with a few more orders of recursion to obtain results fairly close to those that would be obtained upon an infinite recursion, assuming convergence. So let us now look at that.

We can ascertain the general trend toward convergence or divergence simply using the $n=1$ recursive order, because as we have seen, the basic pattern for higher orders is already established at first order. For $W(J)_1$ we have:

$$\begin{aligned}(2\pi)^4 W(J)_1 &= \int d^4kJ_\sigma \pi_1 J^\sigma = \int d^4kJ_\sigma \left(\pi_0{}^{-1} + \pi_0 J_\tau k^\tau + \pi_0 J_\tau \pi_0 J^\tau\right)^{-1} J^\sigma \\ &= \int d^4kJ_\sigma \left(k_\tau k^\tau - m^2 + i\varepsilon + \frac{J_\tau k^\tau}{k_\tau k^\tau - m^2 + i\varepsilon} + \frac{J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} \right)^{-1} J^\sigma \\ &= \int d^4kJ_\sigma \left(\frac{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3 + \left(k_\tau k^\tau - m^2 + i\varepsilon\right) J_\tau k^\tau + J_\tau J^\tau}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} \right)^{-1} J^\sigma\end{aligned} \quad . \tag{11.7}$$

Given $J^\mu = \lambda^i{}_{AB} J^{i\mu} = J^\mu{}_{AB}$ for SU(3)×U(1) has rank 3 at the same time that $\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3 = \delta_{AB}\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3$ sits on the third rank diagonal, a naive look at (11.7) tells us that the dominant term in the numerator will be $\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3$ for $J_\tau k^\tau < \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2$ and $J_\tau J^\tau < \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3$. But when considering the matrix equations, a more precise statement would say that $\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3$ represents eigenvalues of $\delta \equiv \left(k_\tau k^\tau - m^2 + i\varepsilon\right) J_\tau k^\tau + J_\tau J^\tau$, and will dominate when these eigenvalues are larger rather than smaller. In the case where $J_\tau k^\tau$ and $J_\tau J^\tau$ are small and substantially neglectable in relation to $\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2$ and $\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3$, the overall expression (11.7) will be:

$$(2\pi)^4 W(J)_1 = \int d^4kJ_\sigma \left(\frac{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^3 + \delta}{\left(k_\tau k^\tau - m^2 + i\varepsilon\right)^2} \right)^{-1} J^\sigma \cong \int d^4kJ_\sigma \frac{1}{k_\tau k^\tau - m^2 + i\varepsilon} J^\sigma \, , \tag{11.8}$$

which is of the same form as the abelian propagator. So the solution (11.6) would appear to be fully convergent (or, at least no more divergent than the abelian path integral) for $J_\tau k^\tau$ and $J_\tau J^\tau$ which are small in comparison to eigenvalues which are specific powers of $k_\tau k^\tau - m^2 + i\varepsilon$.

Finally, because (11.5) is an exact analytical calculation using a closed recursive kernel, this "prove(s) that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$."

12. Summary and Conclusion

This concludes the formal development of this paper, so let us summarize what we have learned: In non-abelian gauge theory with gauge fields *G*, although the magnetic charge density $P = DF = DDG = 0$ by a Jacobian identity (2.4) just as the abelian magnetic charge density $P = dF = ddG = 0$ because of the differential forms geometry, there is still a non-vanishing magnetic field flux $\oiint F = -i\oiint [G,G] = -i\iiint dGG \neq 0$ (3.3) across closed surfaces which contrasts to the zero net flux $\oiint F = 0$ that one has in abelian gauge theory. These apparently-conflicting features of non-abelian theory – namely a non-zero magnetic flux over closed surfaces but no magnetic sources – are reconciled by realizing that the magnetic field flux is not sourced $\oiint F(P)$ by any elementary magnetic charge density which is $P = DF = DDG = 0$, but rather is sourced $\oiint F(G)$ by a "faux" magnetic source $P' = -id[G,G] = -idGG$ which arises totally from the gauge fields, $P'(G)$. But real gauge fields do not arise spontaneously. They must be sourced by an electric charge density *J*, and in non-abelian gauge theory, the differential equation which governs this is $*J = D*F = D*DG$. Further, we also know that in Dirac theory, electric charge densities are in turn sourced by fermion wavefunctions ψ via Dirac's $J^\mu = \overline{\psi}\gamma^\mu\psi$. Thus, we now need to set upon obtaining the inverse solution to $*J = D*F = D*DG$ for $G(J)$ to enable us to find $\oiint F(G(J))$ and $\oiint F\left(G\left(J(\psi)\right)\right)$.

So in section 5 we develop the electric source field equation $*J = D*F = D*DG$, and in sections 6 and 7 respectively, we carefully develop the inverse solutions $G(J)$ for massive and massless gauge bosons respectively, paying very close attention to issues involving uniqueness and gauge-invariance and gauge fixing and "contextual gauge fixing" wherein a *mathematical* inverse which is non-unique becomes unique when placed into the *physical* context of a conserved current density. And in section 8 we see how $G(J)$ is not really a solution involving *J* alone, but is a highly-non-linear, recursive function $G(G,J)$ which can be recursed as often as desired, and then turned from $G(G,J)$ into $G(J)$ by setting the perturbation $V = 0$ at any desired order. We also noted how the physical inverse ought not to depend on an arbitrary cutoff of the recursion, but rather, ought rather to be based on the series (8.20) that results from recursing an infinite number of times before zeroing the perturbation.

So starting in section 9 we made use of the non-abelian solution for a massive gauge boson, namely $G_\mu = \left(-V + k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ of (6.27) to write out $\oiint F\left(G(G,J)\right)$ in (9.2). Then to keep the initial development simple and develop the "ground state" symmetries, we immediately set $V = 0$ in (9.4) to write $\oiint F\left(G(J)\right)$ in the zeroth recursive order $\left((0)\right)_0$, which is the same thing as having used the abelian massive solution $G_\mu = \left(k_\tau k^\tau - m^2 + i\varepsilon\right)^{-1} J_\mu$ (6.17) a.k.a. (6.28). After then using $J^\mu = \overline{\Psi}\gamma^\mu\Psi$ to replace currents with fermions and thus arrive at $\oiint F\left(G\left(J\left(\psi\right)\right)\right)$ in (9.7), we turned to the fermion Exclusion Principle of Fermi-Dirac-Pauli.

It is the Exclusion Principle that drives the introduction of a rank-3 gauge group to ensure that all of the fermions within the $\oiint F\left(G\left(J\left(\psi\right)\right)\right)$ system are in three distinct eigenstates, turning this now into $\oiint F\left(G\left(J\left(\psi_R,\psi_G,\psi_B\right)\right)\right)$ which reaches the goal established at the end of section 3. The reason for there being three colored quarks in the ground state of a baryon is then seen to be very simple: because there are three additive terms in the covariant tensor expression (9.7) for a magnetic monopole. This also brings with it, eight bi-colored gauge fields. After applying a Goldstone-like mechanism (9.15) to reallocate degrees of freedom and force the gauge fields to be massless and give mass to the fermions while contextually-preserving the uniqueness of the underlying solution for $\left(G\left(J\right)\right)$, we arrive at the ground state monopole density of (9.21). This monopole has the antisymmetric $R[G,B]+G[B,R]+B[R,G]$ color-neutral wavefunction of a baryon although it does also contain fermions in three colored eigenstates, and as we had already found in (3.5), it permits no net flux of individual gauge fields across its closed surfaces. But then we find in (10.4) and (10.5) that this monopole does permit a net flux only of color-neutral $\overline{R}R+\overline{G}G+\overline{B}B$ mesons, which further cements the confinement of gauge fields first suspected in section 3 because nothing *other than* colorless $\overline{R}R+\overline{G}G+\overline{B}B$ fields are permitted to net flow in across closed surfaces. And we further find from (10.6) that the rank-3 gauge group must be SU(3)×U(1), not just SU(3), and that this provides the magnetic monopoles with topological stability so long as this SU(3)×U(1) group emerges following the spontaneous symmetry breaking of a larger simple group $G \supset \mathrm{SU(3)}\times\mathrm{U(1)}$. We learn at (10.9) that the U(1) generator provides a natural platform for equipping each fermion with a baryon number $B=\frac{1}{3}$ and the overall monopole with $B=1$, which now introduces *flavor* to these color-neutral monopoles and mesons and their colored fermions and gauge bosons. And we see in (10.10) and (10.11) that one can thereafter arrive at suitable generator assignments which give rise to the correct electric charges $Q=+1$ for the proton and by a disconnected assignment (which then requires a larger unifying group) $Q=0$ for the neutron, as well as the $Q=+\frac{2}{3}$ for the up and $Q=-\frac{1}{3}$ for the down flavors of quark.

Although nuclear and particle physics are often discussed as if they are one and the same discipline, in fact, they are very distinct based on present understandings of each. This fault line which separates nuclear and hadron physics from particle physics is concisely captured by Jaffe and Witten when they state at page 3 of the "Yang-Mills and Mass Gap" problem [6] that:

> ". . . for QCD to describe the strong force successfully . . . It must have 'quark confinement,' that is, even though the theory is described in terms of elementary fields, such as the quark fields, that transform non-trivially under SU(3), the physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant."

It is this difference between "elementary fields, such as the quark [and the gluon] fields, that transform non-trivially under SU(3)" and "the physical particle states—such as the proton, neutron, and pion—[which] are SU(3)-invariant," as well as the need to give flavor to color-neutral baryons and understand the origins of the specific baryon flavors which are protons and neutrons, which separates the elementary particle physics of colored quarks and gluons, from the hadron physics of the colorless baryons and mesons, and the nuclear physics of proton- and neutron-flavored baryons.

As detailed in the discussion following (10.5), if one advances the thesis that the non-abelian faux magnetic monopole of (9.21) is in fact synonymous with a baryon, then the results reviewed in detail in section 10 would appear to solve this *confinement leg of the mass gap problem*, at least in the classical context. Moreover, the results presented here take a critical step forward toward unifying elementary particle physics with hadron physics and nuclear physics. It is equation (9.21) which operates as a "bridge" between the elementary particle physics of colored quarks and gluons and the hadron physics of the colorless baryons and mesons. This is because (9.21), together with its related consequence (10.5), demonstrates how quark and gluon fields that transform non-trivially under SU(3) assemble together into the *colorless*, *SU(3)-invariant particle states* which are baryons and mesons, that is, hadrons. Then, the non-vanishing trace of (9.21) forces us to employ SU(3)×U(1). This ensures topological stability which is required if (9.21) is to be associated with stable physical particles such as the neutron and especially the proton. Further, via the new U(1) generator, this introduces flavor which then allows these baryons to be flavored into the protons and neutrons at the heart of nuclear physics.

Of course, as discussed in section 4 there are many reasons to believe confinement is related to the running of the coupling constant which is an inherently quantum effect. But as also argued in section 4, one might take the perspective that the *cause* for confinement and baryon compositeness is the classical field equation (3.3) for a Yang-Mills monopole which has the symmetry (3.5), and that one of the *effects* of this is that in a quantum field treatment of these baryon monopoles, the strong coupling will weaken for ultraviolet and strengthen for infrared probes. Without more, however, one could fairly conclude that the connections suggested between some identities of the classical Yang-Mills equation and confinement in the quantum theory are simply still too speculative or weakly supported to constitute a viable theory of hadronic physics, especially since quarks are alluded to but not shown to be required.

But sections 9 and 10 overcome any such conclusion. These sections deepen support for the argument made in sections 3 and 4 by demonstrating that a further *cause* for confinement is the color-neutral SU(3)-invariance of both the monopole (9.21) and the meson (10.5), which might then be expected in a quantum field treatment to reveal the *effect* of a running coupling constant which is consistent with these root causes that are *already seen in the classical theory*. It is certainly true that an important view of confinement is the quantum view of a running coupling. But so too is Jaffe and Witten's complementary symmetry view of confinement as utilized here, in which "even though [a] theory is described in terms of elementary fields, such as the quark fields, that transform non-trivially under SU(3), the physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant." Sections 9 and 10 here make clear that Yang-Mills monopoles manifest these required confinement symmetries. And, this underscores the value as argued in section 4, of finding and fleshing out, the right classical theory to quantize, before trying to leap unarmed into quantization. Colloquially speaking, classical theory is the "horse" which must one must precede the "cart" of quantization.

As to the "cart" of quantization, two further points may now be made in light of the development after section 4, to supplement those already made in section 4. First, as noted in section 4, the chiral anomaly provides an object lesson that not every symmetry which appears in a classical theory carries through to the associated quantum theory. As pointed out in section 7, any divergence there may be between classical and quantum symmetries emanates from the measure $D\varphi$ which is the integration variable in the path integral. A classical symmetry exists if some transformation leaves the action $S(\varphi)$ invariant. A quantum symmetry exists (and inherits the classical symmetry) if the same transformation leaves the path integral $Z=\int D\varphi \exp iS(\varphi)$ invariant. So, for example, although the classical monopole (9.21) has the color-neutral baryon wavefunction $R[G,B]+G[B,R]+B[R,G]$ and the classical net-flowing magnetic field (10.5) has the color-neutral meson wavefunction $\overline{R}R+\overline{G}G+\overline{B}B$, i.e., are classically invariant under an SU(3) gauge transformation, it is valid to ask whether these symmetries will carry through to the related quantum objects. This cannot be answered with absolute certainty until one has the complete quantum theory corresponding to the foregoing classical development, but it is encouraging to note that the observed baryons and the mesons of *quantum* physics are also known to be color-neutral with the same respective $R[G,B]+G[B,R]+B[R,G]$ and $\overline{R}R+\overline{G}G+\overline{B}B$ wavefunctions. Thus for example, when Jaffe and Witten state on page 3 of [6] that "the physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant," they are not qualifying or restricting this statement to classical particles. *QCD is a quantum theory, and the invariance of baryons and mesons, i.e., hadrons, under SU(3) is a well-known feature not only of classical, but of quantum, chromodynamics. That these symmetries appear to emerge very naturally and inexorably from classical Yang-Mills theory without having to make any separate postulates about SU(3) being a theory of strong interactions, is highly compelling.*

Second, the most important result pertaining to quantization in this paper, is the finding in section 8 and its application in section 11 that the inverse solution $G(J)$ is actually a recursive solution for $G(G,J)$, but that this can be turned into a $G(J)$ solution by recursing to any desired order and then setting the perturbation $V=0$. This is important because, referring to page 6 of [6], the difficulty of being able to:

> "Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ . . ."

is not a physics problem, it is a *mathematics* problem, and more particularly, it is a *calculation* problem of not knowing how to perform an exact analytical calculation of the quantum path integral for Yang-Mills theory in particular, and for non-linear physics theories in general.

Specifically, as discussed in section 8, the technique of analytically calculating a path integral $Z=\int DG \exp iS(G)=\mathscr{C}\exp iW(J)$ revolves around clever extrapolations of the

Gaussian integral $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right) = \left(-2\pi / A\right)^{.5} \exp\left(J^2 / 2A\right)$ which only contains *x* and x^2 and no higher order in the integration variable *x*. Put an x^3 or an x^4 into this integral, or even worse, put any higher-order polynomial into this integral, and it is simply not known mathematically how to calculate this integral at all. So the *physics* recipe for quantizing Yang-Mills is very clear: find the action, and use it in a path integral. But the mathematical technique for how to calculate this is not known. The best anybody had been able to do thus far is to make use of (8.25) to replace gauge fields with $G_\mu \to \delta / \delta J^\mu$ and then remove $\exp\left(V\left(\delta / \delta J\right)\right)$ from the integral so all that remains behind to integrate is the simple $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right)$. Generally speaking, we need to replace the gauge fields *G* with current densities *J*, and leave behind the simple quadratic form $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right)$. What we find in section 8 is a new and different way to make a $G \to J$ substitution in lieu of the usual $G_\mu \to \delta / \delta J^\mu$: recurse $G_\mu = \left(k_\tau k^\tau - m^2 + i\varepsilon + G_\tau k^\tau + G_\tau G^\tau\right)^{-1} J_\mu$ to any desired order, then set $V = -G_\tau k^\tau - G_\tau G^\tau$ (because $k_\tau G^\tau = 0$) to zero so that all gauge fields are removed. By recursing to infinite order and removing these gauge fields, we can arrive at an expression for $G(J)$ with all the gauge fields removed, and be left with only having to integrate $\int dx \exp\left(-\frac{1}{2}Ax^2 - Jx\right)$. In short, the recursion preliminarily developed in section 8 provides the needed *mathematical tools* to carry out exact analytical calculations of what are now seemingly-intractable path integrations for non-linear *physical* field theories.

In section 11, we then show how to apply these recursive results to calculate the non-linear Yang-Mills path integral over the gauge field portion DG of the path integral measure analytically and exactly, thereby proving the existence of a non-trivial relativistic quantum Yang–Mills theory exists on $\mathbb{R}^4$ for any compact simple gauge group G by solving a mathematical challenge for which the solution has not previously been known. Having used recursive technique to prove a quantum field theory for Yang-Mills, the question now arises whether recursive technique may be similarly applied to other non-linear field theories, most notably, gravitation.

A next important step is to see if this can be connected to numerically-precise empirical observations relating to protons and neutrons. Among the important unexplained data that we already know about for protons and neutrons are their masses, as well as their binding energies in a wide variety of nuclei. Thus, it becomes important to calculate energies and as pointed out at (10.13), the way to do so is to use (10.13) in the general energy formulation $E = -\iiint \mathfrak{L}_{\text{gauge}} d^3x$, using a combination of $\mathrm{Tr}\left(F_{\sigma\tau}F^{\sigma\tau}\right)$ inner and $\mathrm{Tr}F_{\sigma\tau}\mathrm{Tr}F^{\sigma\tau}$ outer product terms. While we do not do so in this paper, the author has done so before, and published these results in [15], [16] and [21]. Beyond the clear symmetry concurrences developed in section 10, these empirical concurrences provide compelling experimental support for the concluding that the non-zero faux magnetic source densities $P' = -id\left[G, G\right] = -idGG$ are baryon densities, that $\iiint P'$ is a baryon, that $F_{\text{eff}\,\mu\nu} = -i\left[G_\mu, G_\nu\right]$ in (10.4) is a meson field, and that the $\oiint F \neq 0$ which originally actuated this whole line of development represents the interaction of these baryons via mesons, and indeed the nuclear interaction protons and neutrons at classical level. As discussed, although these symmetries were all developed using the classical theory, there is no apparent reason why these symmetries would be lost in the $DGDcDc^\dagger$ measure of the complete path integral $Z = \int DGDcDc^\dagger \exp\left(i\left[S(G) - (1/2\xi)\int d^4x(\partial G)^2\right] + S(c^\dagger, c)\right)$ and would not carry over to the quantum field theory.

In fact, it is well known that the same color symmetries which have been classically developed in the present treatment solely emergent from classical Yang-Mills theory, do carry over to Quantum Chromodynamics.

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