THE TRIANGULAR PROPERTIES OF ASSOCIATED LEGENDRE FUNCTIONS USING THE VECTORIAL ADDITION THEOREM FOR SPHERICAL HARMONICS

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ABSTRACT

Triangular properties of associated Legendre functions are derived using the Vectorial Addition Theorem of spherical harmonics

1. Introduction

Triangular properties of associated Legendre functions were first introduced in reference [1]. They relate associated Legendre functions with the arguments being the cosines of angles in a triangle. They can be used to simplify the calculations of corss sections of electron-atom collisions. They were also encountered in the analytical evaluation of infinte integrals over spherical Bessel functions [2]. This paper arrives at the same result of reference [1] using the Vectorial Addition Theorem of spherical harmonics. A new relation involving a double sum over associated Legendre functions is found using the same technique.

2. Deriving the Triangular Properties

Consider a triangle of sides k_1, k_2 and k_3 such that $\vec{k}_3 = \vec{k}_1 + \vec{k}_2$. Application of the Vectorial Addition Theorem for spherical harmonics [3] results in

$$Y_{\lambda_3}^{M_3}(\hat{k}_3) = (-1)^{\lambda_3 - M_3}\,(2\lambda_3+1)\left(\frac{k_1}{k_3}\right)^{\lambda_3}\sum_{\lambda=0}^{\lambda_3}\sqrt{\frac{4\pi}{(2\lambda+1)[2(\lambda_3-\lambda)+1]}}\binom{2\lambda_3}{2\lambda}^{1/2}$$
$$\times\left(\frac{k_2}{k_1}\right)^{\lambda}\sum_{M}\begin{pmatrix}\lambda_3-\lambda & \lambda & \lambda_3\\ M_3-M & M & -M_3\end{pmatrix} Y_{\lambda_3-\lambda}^{M_3-M}(\hat{k}_1)\,Y_{\lambda}^{M}(\hat{k}_2), \tag{2.1}$$

where $-\lambda_3 \le M_3 \le \lambda_3$ and $-\lambda \le M \le \lambda$. Now let the triangle be in a plane belonging to a specific azimuthal angle ϕ, then using

$$Y_L^M(\hat{k}) \;=\; \sqrt{\frac{2L+1}{4\pi}}\,\sqrt{\frac{(L-M)!}{(L+M)!}}\,e^{im\phi}\,P_L^M(\cos\theta_{\hat{k}}), \tag{2.2}$$

one arrives at

$$\begin{aligned} P_{\lambda_3}^{M_3}(\cos\theta_{\hat{k}_3}) \;=\; & (-1)^{\lambda_3-M_3}\sqrt{\frac{(\lambda_3+M_3)!}{(\lambda_3-M_3)!}}\,\sqrt{2\lambda_3+1}\,\left(\frac{k_1}{k_3}\right)^{\lambda_3}\sum_{\lambda=0}^{\lambda_3}\binom{2\lambda_3}{2\lambda}^{1/2}\left(\frac{k_2}{k_1}\right)^{\lambda} \\ & \times\sum_M\sqrt{\frac{(\lambda-M)!\,[\lambda_3-\lambda-(M_3-M)]!}{(\lambda+M)!\,[\lambda_3-\lambda+M_3-M]!}}\begin{pmatrix}\lambda_3-\lambda & \lambda & \lambda_3\\ M_3-M & M & -M_3\end{pmatrix} \\ & \times P_{\lambda_3-\lambda}^{M_3-M}(\cos\theta_{\hat{k}_1})\,P_\lambda^M(\cos\theta_{\hat{k}_2}). \end{aligned} \tag{2.3}$$

It is easy to show that

$$\begin{aligned} & \sqrt{\frac{(j_1+j_2+m)!\,(j_2-m_2)!}{(j_1+j_2-m)!\,(j_2+m_2)!}}\binom{2(j_1+j_2)}{2j_2}^{1/2}\begin{pmatrix} j_1 & j_2 & j_1+j_2\\ m_1 & m_2 & -m\end{pmatrix} \\ & = \frac{(-1)^{j_1-j_2+m}}{\sqrt{2(j_1+j_2)\,+\,1}}\sqrt{\frac{(j_1+m_1)!}{(j_1-m_1)!}}\binom{j_1+j_2+m}{j_2+m_2}, \end{aligned} \tag{2.4}$$

Hence, equation (2.3) reduces to

$$P_{\lambda_3}^{M_3}(\cos\theta_{\hat{k}_3}) \;=\; \left(\frac{k_1}{k_3}\right)^{\lambda_3}\sum_{\lambda=0}^{\lambda_3}\left(\frac{k_2}{k_1}\right)^{\lambda}\sum_M\binom{\lambda_3+M_3}{\lambda+M}P_{\lambda_3-\lambda}^{M_3-M}(\cos\theta_{\hat{k}_1})\,P_\lambda^M(\cos\theta_{\hat{k}_2}). \tag{2.5}$$

Let $\vec{k}_1$ point in the z-direction, as in Fig. 1, where $\cos\theta_{\hat{k}_1} = 1$, $\cos\theta_{\hat{k}_2} = -\cos\gamma$ and $\cos\theta_{\hat{k}_3} = \cos\beta$, then

$$P_{\lambda_3}^{M_3}(\cos\beta) \;=\; (-1)^{M_3}\left(\frac{k_1}{k_3}\right)^{\lambda_3}\sum_{\lambda=0}^{\lambda_3}\left(\frac{-k_2}{k_1}\right)^{\lambda}\binom{\lambda_3+M_3}{\lambda_3-\lambda}P_\lambda^{M_3}(\cos\gamma). \tag{2.6}$$

using

$$P_L^{\mathcal{M}}(-\cos\theta) \;=\; (-1)^{L+\mathcal{M}}\, P_L^{\mathcal{M}}(\cos\theta), \tag{2.7}$$

$$P_L^{\mathcal{M}}(1) \;=\; \delta_{\mathcal{M},0} \tag{2.8}$$

and

$$\binom{\lambda_3 + M_3}{\lambda + M_3} \;=\; \binom{\lambda_3 + M_3}{\lambda_3 - \lambda}. \tag{2.9}$$

An alternative form, which is the result of reference [2] can be obtained if the sum is made over $\mathcal{L} = \lambda_3 - \lambda$ as follows

$$P_{\lambda_3}^{M_3}(\cos\beta) \;=\; (-1)^{M_3} \left(\frac{k_1}{k_3}\right)^{\lambda_3} \sum_{\mathcal{L}=0}^{\lambda_3 - M_3} \left(\frac{-k_2}{k_1}\right)^{\lambda_3 - \mathcal{L}} \binom{\lambda_3 + M_3}{\mathcal{L}} P_{\lambda_3-\mathcal{L}}^{M_3}(\cos\gamma), \tag{2.10}$$

where the sum is now restricted to $\lambda_3 - M_3$ since $P_{\lambda_3-\mathcal{L}}^{M_3}(\cos\gamma)$ vanishes for $M_3 > \lambda_3 - \mathcal{L}$.

Now let $\vec{k}_2$ point along the z-axis, where $\cos\theta_{\hat{k}_2} = 1$, $\cos\theta_{\hat{k}_1} = -\cos\gamma$ and $\cos\theta_{\hat{k}_3} = \cos\alpha$. Equation (2.5) reduces to

$$P_{\lambda_3}^{M_3}(\cos\alpha) \;=\; (-1)^{M_3} \left(\frac{-k_1}{k_3}\right)^{\lambda_3} \sum_{\lambda=0}^{\lambda_3} \left(\frac{-k_2}{k_1}\right)^{\lambda} \binom{\lambda_3 + M_3}{\lambda} P_{\lambda_3-\lambda}^{M_3}(\cos\gamma). \tag{2.11}$$

Also, if $\vec{k}_3$ points in the z-direction, where $\cos\theta_{\hat{k}_3} = 1$, $\cos\theta_{\hat{k}_1} = \cos\beta$ and $\cos\theta_{\hat{k}_2} = \cos\alpha$, equation (2.5) becomes

$$\sum_{\lambda=0}^{\lambda_3} \left(\frac{k_2}{k_1}\right)^{\lambda} \sum_{M} \binom{\lambda_3}{\lambda + M} P_{\lambda_3-\lambda}^{-M}(\cos\beta)\, P_{\lambda}^{M}(\cos\alpha) \;=\; \left(\frac{k_1}{k_3}\right)^{\lambda_3}. \tag{2.12}$$

3. Conclusions

The Vectorial Addition Theorem is a powerful tool in obtaining relationships between associated Legendre functions, $P_L^{\mathcal{M}}$, for $-L \leq \mathcal{M} \leq L$. The goal is to be able to generalise these relations to associated Legendre functions with $-\infty \leq \mathcal{M} \leq L$, where these functions were defined in reference [2]. They relations allow the simplification of expressions obtained in the analytical evaluation of infinite integrals over spherical Bessel functions.

4. References

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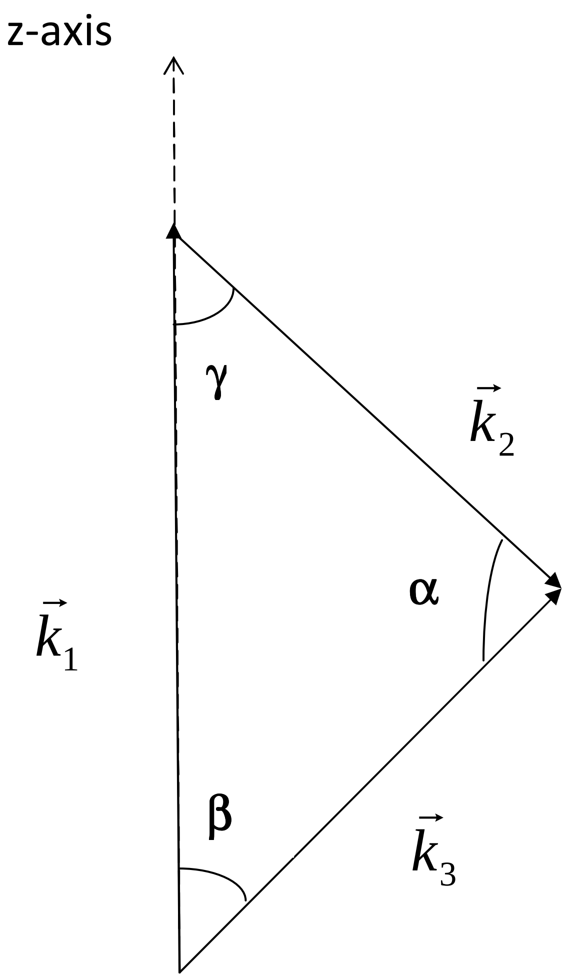

Figure 1: Triangle of sides k_1, k_2 and k_3, where $\vec{k}_1$ points along the z-axis