The function equation $S(n) = Z(n)$

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Abstract For any positive integer $n$, let $S(n)$ and $Z(n)$ denote the Smarandache function and the pseudo Smarandache function respectively. In this paper we prove that the equation $S(n) = Z(n)$ has infinitely many positive integer solutions $n$.

Keywords Smarandache function; Pseudo Smarandache function; Diophantine equation.

For any positive integers $n$, let $S(n)$ and $Z(n)$ denote the Smarandache function and pseudo Smarandache function respectively. In [1], Ashbacher proposed two problems concerning the equation

$$S(n) = Z(n)$$

as follows.

Problem 1. Prove that if $n$ is an even perfect number, then $n$ satisfies (1).

Problem 2. Prove that (1) has infinitely many positive integer solutions $n$.

In this paper we completely solve these problems as follows.

Theorem 1. If $n$ is an even perfect number, then (1) holds.

Theorem 2. (1) has infinitely many positive integer solutions $n$.

Proof of Theorem 1. By [2, Theorem 277], if $n$ is an even perfect number, then

$$n = 2^{p-1}(2^p - 1), \quad (2)$$

where $p$ is a prime. By [3] and [4], we have

$$S(n) = 2^p - 1. \quad (3)$$

On the other hand, since

$$\frac{1}{2} (2^p - 1) ((2^p - 1) + 1) = n, \quad (4)$$

by (2), we get

$$Z(n) = 2^p - 1 \quad (5)$$

immediately. The combination of (3) and (5) yields (1). Thus, the theorem is proved.

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Proof of Theorem 2. Let \( p \) be an odd prime with \( p \equiv 3 \pmod{4} \). Since \( S(2) = 2 \) and \( S(p) = p \), we have

\[
S(2p) = \max(S(2), S(p)) = \max(2, p) = p.
\] (6)

Let \( t = Z(2p) \), By the define of \( Z(n) \), we have

\[
\frac{1}{2}(t + 1) \equiv 0 \pmod{2p}.
\] (7)

It implies that either \( t \equiv 0 \pmod{p} \) or \( t + 1 \equiv 0 \pmod{p} \). Hence, we get \( t \geq p - 1 \). If \( t = p - 1 \), then from (7) we obtain

\[
\frac{1}{2}(p - 1)p \equiv 0 \pmod{2p}.
\] (8)

whence we get

\[
\frac{1}{2}(p - 1)p \equiv 0 \pmod{2}.
\] (9)

But, since \( p \equiv 3 \pmod{4} \), (9) is impossible. So we have

\[ t \geq p. \] (10)

Since \( p + 1 \equiv 0 \pmod{4} \), we get

\[
\frac{1}{2}b(p + 1) \equiv 0 \pmod{2p}
\] (11)

and \( t = p \) by (10). Therefore, by (6), \( n = 2p \) is a solution of (1). Notice that there exist infinitely many primes \( p \) with \( p \equiv 3 \pmod{4} \). It implies that (1) has infinitely many positive integer solutions \( n \). The theorem is proved.

References
