ON THE SMARANDCHE FUNCTION AND ITS HYBRID MEAN VALUE

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Abstract For any positive integer $n$, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined as the smallest $m \in N^+$ with $n|\mu(m)$. In this paper, we study the asymptotic property of a hybrid mean value of the Smarandache function and the Mangoldt function, and give an interesting hybrid mean value formula for it.

Keywords: the Smarandche function; the Mangoldt function; Mean value.

§1. Introduction

For any positive integer $n$, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined as the smallest $m \in N^+$ with $n|\mu(m)$. From the definition of $S(n)$, one can easily deduce that if $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is the prime power factorization of $n$, then

$$S(n) = \max_{1 \leq i \leq k} S(p_i^{a_i}).$$

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [2]). In this paper, we study the asymptotic property of a hybrid mean value of the Smarandache function and the Mangoldt function, and give an interesting hybrid mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} \mu(n)S(n) = \frac{x^2}{x} + O \left( \frac{x^2 \log \log x}{\log x} \right),$$

where $\mu(n)$ is the Mangoldt function defined by

$$\mu(n) = \begin{cases} \log p, & \text{if } n = p^\alpha (\alpha \geq 1); \\ 0, & \text{otherwise.} \end{cases}$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Firstly, we need following:
Lemma. For any prime $p$ and any positive integer $\alpha$, we have

$$S(p^\alpha) = (p - 1)\alpha + O \left( \frac{p \log \alpha}{\log p} \right).$$

Proof. From Theorem 1.4 of reference [3], we can obtain the estimate. Now we use the above Lemma to complete the proof of the theorem. From the definition of $\wedge(n)$, we have

$$\sum_{n \leq x} \wedge(n) S(n) = \sum_{p^\alpha \leq x} S(p^\alpha) \log p$$

$$= \sum_{p \leq x} \sum_{\alpha \leq \frac{\log x}{\log p}} \log p \left( (p - 1)\alpha + O \left( \frac{p \log \alpha}{\log p} \right) \right)$$

$$= \sum_{p \leq x} (p - 1) \log p \sum_{\alpha \leq \frac{\log x}{\log p}} \alpha + O \left( \sum_{p \leq x} \sum_{\alpha \leq \frac{\log x}{\log p}} \log \alpha \right).$$

Applying Euler’s summation formula, we can get

$$\sum_{\alpha \leq \frac{\log x}{\log p}} \alpha = \frac{1}{2} \frac{\log^2 x}{\log^2 p} + O \left( \frac{\log x}{\log p} \right),$$

and

$$\sum_{\alpha \leq \frac{\log x}{\log p}} \log \alpha = \frac{\log x}{\log p} \log \frac{\log x}{\log p} - \frac{\log x}{\log p} + O \left( \frac{\log x}{\log p} \right).$$

Therefore we have

$$\sum_{n \leq x} \wedge(n) S(n) = \frac{1}{2} \log^2 x \sum_{p \leq x} \frac{p}{\log p} + O \left( \log x \log \log x \sum_{p \leq x} \frac{p}{\log p} \right). \quad (1)$$

If $x > 0$ let $\pi(x)$ denote the number of primes not exceeding $x$, and let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is a prime;} \\ 0, & \text{otherwise.} \end{cases}$$

then $\pi(x) = \sum_{p \leq x} a(n)$. Note the asymptotic formula

$$\pi(x) = \frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right).$$
and from Abel’s identity, we have

\[
\begin{align*}
\sum_{p \leq x} \frac{p}{\log p} &= \sum_{n \leq x} a(n) \frac{n}{\log n} \\
&= \pi(x) \frac{x}{\log x} - \pi(2) \frac{2}{\log 2} - \int_2^x \pi(t) \frac{t}{\log t} \, dt \\
&= \frac{x}{\log x} \left( \frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right) \right) - \int_2^x \left( \frac{t}{\log t} + O \left( \frac{t}{\log^2 t} \right) \right) \, dt \\
&= \frac{1}{2} \frac{x^2}{\log^2 x} + O \left( \frac{x^2}{\log^3 x} \right). \quad (2)
\end{align*}
\]

Combining (1) and (2), we have

\[
\sum_{n \leq x} \Lambda(n) S(n) = 1 + O \left( \frac{x^2}{\log x} \right) + O \left( \frac{x^2 \log \log x}{\log x} \right).
\]

This completes the proof of the theorem.

References


