On a conjecture involving the function $SL^*(n)$

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Abstract In this paper, we define a new arithmetical function $SL^*(n)$, which is related with the famous F.Smarandache LCM function $SL(n)$. Then we studied the properties of $SL^*(n)$, and solved a conjecture involving function $SL^*(n)$.

Keywords F.Smarandache LCM function, $SL^*(n)$ function, conjecture.

§1. Introduction and result

For any positive integer $n$, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer $k$ such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of all positive integers from 1 to $k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $\cdots$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of $n$ into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r}\}.$$ 

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [2], [4] and [5]. For example, Murthy [2] proved that if $n$ be a prime, then $SL(n) = S(n)$, where $S(n)$ be the F.Smarandache function. That is, $S(n) = \min\{m : n \mid m!, m \in N\}$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n ? \quad (1)$$

Le Maohua [4] solved this problem completely, and proved the following conclusion:

Every positive integer $n$ satisfying (1) can be expressed as

$$n = 12 \quad \text{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_1^{\alpha_1}, i = 1, 2, \cdots, r$. 


Zhongtian Lv [5] proved that for any real number \( x > 1 \) and fixed positive integer \( k \), we have the asymptotic formula

\[
\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} c_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),
\]

where \( c_i \) (\( i = 2, 3, \ldots, k \)) are computable constants.

Now, we define another function \( SL^*(n) \) as follows: \( SL^*(1) = 1 \), and if \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the factorization of \( n \) into primes powers, then

\[
SL^*(n) = \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\},
\]

where \( p_1 < p_2 < \cdots < p_r \) are primes.

About the elementary properties of function \( SL^*(n) \), it seems that none has studied it yet, at least we have not seen such a paper before. It is clear that function \( SL^*(n) \) is the dual function of \( SL(n) \). So it has close relations with \( SL(n) \). In this paper, we use the elementary method to study the following problem: For any positive integer \( n \), whether the summation

\[
\sum_{d|n} \frac{1}{SL^*(n)}
\]

is a positive integer? where \( \sum_{d|n} \) denotes the summation over all positive divisors of \( n \).

We conjecture that there is no any positive integer \( n > 1 \) such that (2) is an integer. In this paper, we solved this conjecture, and proved the following:

**Theorem.** There is no any positive integer \( n > 1 \) such that (2) is an positive integer.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem directly. For any positive integer \( n > 1 \), let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the factorization of \( n \) into primes powers, from the definition of \( SL^*(n) \) we know that

\[
SL^*(n) = \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\}.
\]

Now if \( SL(n) = p_k^{\alpha_k} \) (where \( 1 \leq k \leq r \)) and \( n \) satisfy

\[
\sum_{d|m} \frac{1}{SL^*(d)} = N, \quad \text{a positive integer},
\]

then let \( n = m \cdot p_k^{\alpha_k} \) with \( (m, p_k) = 1 \), note that for any \( d|m \) with \( d > 1 \), \( SL^*\left(p_k^{\alpha_k} \cdot d\right) = m \cdot p_k^{\alpha_k-1} \), where \( i = 0, 1, 2, \ldots, \alpha_k \). We have

\[
N = \sum_{d|m} \frac{1}{SL^*(d)} = \sum_{i=0}^{\alpha_k} \sum_{d|m} \frac{1}{SL^*(d \cdot p_k^{i})} = \sum_{i=0}^{\alpha_k} \frac{1}{SL^*(p_k^{i})} + \sum_{i=0}^{\alpha_k} \sum_{d|m} \frac{1}{SL^*(d \cdot p_k^{i})}
\]

\[
= 1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k}} + \sum_{i=0}^{\alpha_k} \sum_{d|m} \frac{1}{SL^*(d \cdot p_k^{i})},
\]
or

\[ m \cdot p_k^{\alpha_k - 1} \cdot N = \sum_{i=0}^{\alpha_k - 1} \sum_{d|m, d > 1} \frac{m \cdot p_k^{\alpha_k - 1}}{SL^*(d \cdot p_k^i)} + m \cdot p_k^{\alpha_k - 1} \cdot \left( \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots + \frac{1}{p_k^{\alpha_k - 1}} \right) + \frac{m}{p_k}. \]  

(4)

It is clear that for any \( d|m \) with \( d > 1 \),

\[ \sum_{i=0}^{\alpha_k - 1} \sum_{d|m, d > 1} \frac{m \cdot p_k^{\alpha_k - 1}}{SL^*(d \cdot p_k^i)} \text{ and } m \cdot p_k^{\alpha_k - 1} \cdot \left( \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots + \frac{1}{p_k^{\alpha_k - 1}} \right), \]

are integers, but \( \frac{m}{p_k} \) is not an integer. This contradicts with (4). So the theorem is true. This completes the proof of the theorem.

**Open problem.** If \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the factorization of \( n \) into primes powers, whether there exists an integer \( n \geq 2 \) such that \( \sum_{d|n} \frac{1}{SL(n)} \) is an integer?

**References**


