On Functions Preserving Convergence of Series in Fuzzy n-Normed Spaces

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Abstract: The purpose of this paper is to introduce finite convergence sequences and functions preserving convergence of series in fuzzy n-normed spaces.

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§1. Introduction

A Pseudo-Euclidean space is a particular Smarandache space defined on a Euclidean space \( \mathbb{R}^n \) such that a straight line passing through a point \( p \) may turn an angle \( \theta_p \geq 0 \). If \( \theta_p > 0 \), then \( p \) is called a non-Euclidean point. Otherwise, a Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean. In [7], S. Gähler introduced \( n \)-norms on a linear space. A detailed theory of \( n \)-normed linear space can be found in [9,12,14,15]. In [9], H. Gunawan and M. Mashadi gave a simple way to derive an \((n-1)\)-norm from the \( n \)-norm in such a way that the convergence and completeness in the \( n \)-norm is related to those in the derived \((n-1)\)-norm. A detailed theory of fuzzy normed linear space can be found in [1,2,4,5,6,11,13,18]. In [16], A. Narayanan and S. Vijayabalaji have extended the \( n \)-normed linear space to fuzzy \( n \)-normed linear space and in [20] the authors have studied the completeness of fuzzy \( n \)-normed spaces.

The main purpose of this paper is to study the results concerning infinite series (see, [3,17,19,21]) in fuzzy \( n \)-normed spaces. In section 2, we quote some basic definitions of fuzzy \( n \)-normed spaces. In section 3, we consider the absolutely convergent series in fuzzy \( n \)-normed spaces and obtain some results on it. In section 4, we study the property of finite convergence sequences in fuzzy \( n \)-normed spaces. In the last section we introduce and study the concept of

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function preserving convergence of series in fuzzy \( n \)-norm spaces and obtain some results.

\section{Preliminaries}

Let \( n \) be a positive integer, and let \( X \) be a real vector space of dimension at least \( n \). We recall the definitions of an \( n \)-seminorm and a fuzzy \( n \)-norm [16].

\textbf{Definition 2.1} A function \((x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|\) from \( X^n \) to \([0, \infty)\) is called an \( n \)-seminorm on \( X \) if it has the following four properties:

\begin{enumerate}
  \item \( \|x_1, x_2, \ldots, x_n\| = 0 \) if \( x_1, x_2, \ldots, x_n \) are linearly dependent;
  \item \( \|x_1, x_2, \ldots, x_n\| \) is invariant under any permutation of \( x_1, x_2, \ldots, x_n \);
  \item \( \|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, \ldots, x_{n-1}, x_n\| \) for any real \( c \);
  \item \( \|x_1, \ldots, x_{n-1}, y + z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\| \).
\end{enumerate}

An \( n \)-seminorm is called a \( n \)-norm if \( \|x_1, x_2, \ldots, x_n\| > 0 \) whenever \( x_1, x_2, \ldots, x_n \) are linearly independent.

\textbf{Definition 2.2} A fuzzy subset \( N \) of \( X^n \times \mathbb{R} \) is called a fuzzy \( n \)-norm on \( X \) if and only if:

\begin{enumerate}
  \item \( N(x_1, x_2, \ldots, x_n, t) = 0 \) for all \( t \leq 0 \);
  \item \( N(x_1, x_2, \ldots, x_n, t) = 1 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent;
  \item \( N(x_1, x_2, \ldots, x_n, t) \) is invariant under any permutation of \( x_1, x_2, \ldots, x_n \);
  \item \( N(x_1, x_2, \ldots, x_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|}) \);
  \item \( N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \min \{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t)\} \).
\end{enumerate}

The pair \((X, N)\) will be called a \textit{fuzzy \( n \)-normed space}.

\textbf{Theorem 2.1} Let \( \mathcal{A} \) be the family of all finite and nonempty subsets of fuzzy \( n \)-normed space \((X, N)\) and \( A \in \mathcal{A} \). Then the system of neighborhoods

\[ \mathcal{B} = \{B(t, r, A) : t > 0, 0 < r < 1, A \in \mathcal{A}\} \]
where \( B(t,r,A) = \{ x \in X : N(a_1, \ldots, a_{n-1}, x, t) > 1 - r, a_1, \ldots, a_{n-1} \in A \} \) is a base of the null vector \( \theta \), for a linear topology on \( X \), named \( N \)-topology generated by the fuzzy \( n \)-norm \( N \).

**Proof** We omit the proof since it is similar to the proof of Theorem 3.6 in [8]. \( \square \)

**Definition 2.3** A sequence \( \{x_k\} \) in a fuzzy \( n \)-normed space \((X, N)\) is said to converge to \( x \) if given \( r > 0, t > 0, 0 < r < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_k - x, t) > 1 - r \) for all \( k \geq n_0 \).

**Definition 2.4** A sequence \( \{x_k\} \) in a fuzzy \( n \)-normed space \((X, N)\) is said to be Cauchy sequence if given \( \epsilon > 0, t > 0, 0 < \epsilon < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_m - x_k, t) > 1 - \epsilon \) for all \( m, k \geq n_0 \).

**Theorem 2.1** ([13]) Let \( N \) be a fuzzy \( n \)-norm on \( X \). Define for \( x_1, x_2, \ldots, x_n \in X \) and \( \alpha \in (0,1) \)

\[
\|x_1, x_2, \ldots, x_n\|_\alpha = \inf \{ t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha \}.
\]

Then the following statements hold.

\( (A_1) \) for every \( \alpha \in (0,1) \), \( \| \cdot, \cdot, \ldots, \cdot \|_\alpha \) is an \( n \)-seminorm on \( X \);

\( (A_2) \) If \( 0 < \alpha < \beta < 1 \) and \( x_1, x_2, \ldots, x_n \in X \) then

\[
\|x_1, x_2, \ldots, x_n\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\beta.
\]

**Example 2.3** ([10, Example 2.3]) Let \( \| \cdot, \cdot, \ldots, \cdot \| \) be a \( n \)-norm on \( X \). Then define \( N(x_1, x_2, \ldots, x_n, t) = 0 \) if \( t \leq 0 \) and, for \( t > 0 \),

\[
N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.
\]

Then the seminorms (2.1) are given by

\[
\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1 - \alpha \|x_1, x_2, \ldots, x_n\|}.
\]

§3. **Absolutely Convergent Series in Fuzzy \( n \)-Normed Spaces**

In this section we introduce the notion of the absolutely convergent series in a fuzzy \( n \)-normed space \((X, N)\) and give some results on it.

**Definition 3.1** The series \( \sum_{k=1}^{\infty} x_k \) is called absolutely convergent in \((X, N)\) if

\[
\sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \infty
\]

for all \( a_1, \ldots, a_{n-1} \in X \) and all \( \alpha \in (0,1) \).
Using the definition of $\|\ldots\|_\alpha$ the following lemma shows that we can express this condition directly in terms of $N$.

**Lemma 3.1** The series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent in $(X, N)$ if, for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$ there are $t_k \geq 0$ such that $\sum_{k=1}^{\infty} t_k < \infty$ and $N(a_1, \ldots, a_{n-1}, x_k, t_k) \geq \alpha$ for all $k$.

**proof** Let $\sum_{k=1}^{\infty} x_k$ be absolutely convergent in $(X, N)$. Then

$$\sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \infty$$

for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$. Let $a_1, \ldots, a_{n-1} \in X$ and $\alpha \in (0, 1)$. For every $k$ there is $t_k \geq 0$ such that $N(a_1, \ldots, a_{n-1}, x_k, t_k) \geq \alpha$ and

$$t_k < \|a_1, \ldots, a_{n-1}, x_k\|_\alpha + \frac{1}{2^k}.$$

Then

$$\sum_{k=1}^{\infty} t_k < \sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

The other direction is even easier to show. \hfill \Box

**Definition 3.2** A fuzzy $n$–normed space $(X, N)$ is said to be sequentially complete if every Cauchy sequence in it is convergent.

**Lemma 3.2** Let $(X, N)$ be sequentially complete, then every absolutely convergent series $\sum_{k=1}^{\infty} x_k$ converges and

$$\left\|a_1, \ldots, a_{n-1}, \sum_{k=1}^{\infty} x_k\right\|_\alpha \leq \sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha$$

for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$.

**Proof** Let $\sum_{k=1}^{\infty} x_k$ be an infinite series such that $\sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \infty$ for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$. Let $y_n = \sum_{k=1}^{n} x_k$ be a partial sum of the series. Let $n = N + 1$ such that $\sum_{k=N+1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \epsilon$. 

Then, for \( n > m \geq N \),
\[
\left\| a_1, \ldots, a_{n-1}, y_n \right\|_\alpha - \left\| a_1, \ldots, a_{n-1}, y_m \right\|_\alpha 
\leq \sum_{k=m+1}^{n} \| a_1, \ldots, a_{n-1}, x_k \|_\alpha 
\leq \sum_{k=N+1}^{\infty} \| a_1, \ldots, a_{n-1}, x_k \|_\alpha 
< \epsilon.
\]

This is shows that \( \{ y_n \} \) is a Cauchy sequence in \((X, N)\). But since \((X, N)\) is sequentially complete, the sequence \( \{ y_n \} \) converges and so the series \( \sum_{k=1}^{\infty} x_k \) converges. \( \square \)

**Definition 3.3** Let \( I \) be any denumerable set. We say that the family \((x_\alpha)_{\alpha \in I}\) of elements in a complete fuzzy \( n \)-normed space \((X, N)\) is absolutely summable, if for a bijection \( \Psi \) of \( \mathbb{N} \) (the set of all natural numbers) onto \( I \) the series \( \sum_{n=1}^{\infty} x_{\Psi(n)} \) is absolutely convergent.

The following result may not be surprising but the proof requires some care.

**Theorem 3.1** Let \((x_\alpha)_{\alpha \in I}\) be an absolutely summable family of elements in a sequentially complete fuzzy \( n \)-normed space \((X, N)\). Let \((B_n)\) be an infinite sequence of a non-empty subset of \( A \), such that \( A = \bigcup_{n} B_n \), \( B_i \cap B_j = \emptyset \) for \( i \neq j \), then if \( z_n = \sum_{\alpha \in B_n} x_\alpha \), the series \( \sum_{n=0}^{\infty} z_n \) is absolutely convergent and \( \sum_{n=0}^{\infty} z_n = \sum_{\alpha \in I} x_\alpha \).

**Proof** It is easy to see that this is true for finite disjoint unions \( I = \bigcup_{n=1}^{N} B_n \). Now consider the disjoint unions \( I = \bigcup_{n=1}^{\infty} B_n \). By Lemma 3.2
\[
\sum_{n=1}^{\infty} \left\| a_1, \ldots, a_{n-1}, z_n \right\|_\alpha 
\leq \sum_{n=1}^{\infty} \sum_{i \in B_n} \left\| a_1, \ldots, a_{n-1}, x_i \right\|_\alpha 
= \sum_{i \in I} \left\| a_1, \ldots, a_{n-1}, x_i \right\|_\alpha < \infty
\]
for every \( a_1, \ldots, a_{n-1} \in X \), and every \( \alpha \in (0, 1) \). Therefore, \( \sum_{n=0}^{\infty} z_n \) is absolutely convergent. Let \( y = \sum_{i \in I} x_i \), \( z = \sum_{n=1}^{\infty} z_n \). Let \( \epsilon > 0 \), \( a_1, \ldots, a_{n-1} \in X \) and \( \alpha \in (0, 1) \). There is a finite set \( J \subset I \) such that
\[
\sum_{i \notin J} \left\| a_1, \ldots, a_{n-1}, x_i \right\|_\alpha < \frac{\epsilon}{2}.
\]
Choose \( N \) large enough such that \( B = \bigcup_{n=1}^{N} B_n \supset J \) and
\[
\left\| a_1, \ldots, a_{n-1}, z - \sum_{n=1}^{N} z_n \right\|_\alpha < \frac{\epsilon}{2}.
\]
Therefore,

\[ \left\| a_1, \ldots, a_{n-1}, y - \sum_{i \in B} x_i \right\|_\alpha < \frac{\epsilon}{2}. \]

By the first part of the proof

\[ \sum_{n=1}^{N} \| z_n \| = \sum_{i \in B} \| x_i \|. \]

Therefore, \( \| a_1, \ldots, a_{n-1}, y - z \|_\alpha < \epsilon. \) This is true for all \( \epsilon \) so \( \| a_1, \ldots, a_{n-1}, y - z \|_\alpha = 0. \) This is true for all \( a_1, \ldots, a_{n-1} \in X, \) \( \alpha \in (0, 1) \) and \( (X, N) \) is Hausdorff see [8, Theorem 3.1]. Hence \( y = z. \)

**Definition 3.4** Let \( (X^*, N) \) be the dual of fuzzy \( n \)-normed space \( (X, N) \). A linear functional \( f : X^* \to K \) where \( K \) is a scalar field of \( X \) is said to be bounded linear operator if there exists a \( \lambda > 0 \) such that

\[ \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha \leq \lambda \| a_1, \ldots, a_{n-1}, x_k \|_\alpha, \]

for all \( a_1, \ldots, a_{n-1} \in X \) and all \( \alpha \in (0, 1). \)

**Definition 3.5** The series \( \sum_{k=1}^{\infty} x_k \) is said to be weakly absolutely convergent in \( (X, N) \) if

\[ \sum_{k=1}^{\infty} \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha < \infty \]

for all \( f \in X^* \), all \( a_1, \ldots, a_{n-1} \in X \) and all \( \alpha \in (0, 1). \)

**Theorem 3.2** Let the series \( \sum_{k=1}^{\infty} x_k \) be weakly absolutely convergence in \( (X, N) \). Then there exists a constant \( \lambda > 0 \) such that

\[ \sum_{k=1}^{\infty} \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha \leq \lambda \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha \]

**Proof** Let \( \{ e_r \}_{r=1}^{\infty} \) be a standard basis of the space \( (X, N) \). Define continuous operators \( S_r : X^* \to X \) where \( r \in \mathbb{Z} \) by the formula \( S_r(f) = \sum_{k=1}^{r} f(x_k)e_k, \) we have

\[ \| a_1, \ldots, a_{n-1}, S_r(f) \|_\alpha = \sum_{k=1}^{r} \| a_1, \ldots, a_{n-1}, f(x_k)e_k \|_\alpha. \]

Since for any fixed \( f \in X^* \), the numbers \( \| a_1, \ldots, a_{n-1}, S_r(f) \|_\alpha \) are bounded by \( \sum_{k=1}^{\infty} \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha, \) by Banach-Steinhaus theorem, we have

\[ \sup_r \| a_1, \ldots, a_{n-1}, S_r(f) \|_\alpha = \lambda < \infty. \]

Therefore,

\[ \sum_{k=1}^{\infty} \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha = \sup_r \| a_1, \ldots, a_{n-1}, S_r(f) \|_\alpha \leq \lambda \| a_1, \ldots, a_{n-1}, f(x_k) \|_\alpha. \]
§4. Finite Convergent Sequences in Fuzzy $n$-Normed Spaces

In this section our principal goal is to show that every sequence having finite convergent property is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property in every metrizable fuzzy $n$-normed space $(X, N)$.

**Definition 4.1** A sequence $\{x_k\}$ in a fuzzy $n$-normed space $(X, N)$ is said to have finite convergent property if

$$
\sum_{j=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_j - x_{j-1}\|_\alpha < \infty
$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$.

**Definition 4.2** A fuzzy $n$-normed space $(X, N)$ is said to be metrizable, if there is a metric $d$ which generates the topology of the space.

**Theorem 4.1** Let $(X, N)$ be a metrizable fuzzy $n$-normed space, then every sequence having finite convergent property is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property.

*Proof* Since $X$ is metrizable, there is a sequence $\{\|a_{1,r}, \ldots, a_{n-1,r}, x\|_{\alpha_r}\}$ for all $a_{1,r}, \ldots, a_{n-1,r} \in X$ and all $\alpha_r \in (0, 1)$ generating the topology of $X$. We choose an increasing sequence $\{m_{k,1}\}$ such that

$$
\sum_{k=1}^{\infty} \|a_{1,1}, \ldots, a_{n-1,1}, x_{m_{k,1}+1} - x_{m_{k,1}}\|_{\alpha_1} < \infty
$$

where $a_{1,1}, \ldots, a_{n-1,1} \in X$ and $\alpha_1 \in (0, 1)$. Then we choose a subsequence $m_{k,2}$ of $m_{k,1}$ such that

$$
\sum_{k=1}^{\infty} \|a_{1,2}, \ldots, a_{n-1,2}, x_{m_{k,2}+1} - x_{m_{k,2}}\|_{\alpha_2} < \infty
$$

where $a_{1,2}, \ldots, a_{n-1,2} \in X$ and $\alpha_2 \in (0, 1)$. Continuing in this way we construct recursively sequences $m_{k,r}$ such that $m_{k,r+1}$ is a subsequence of $m_{k,r}$ and such that

$$
\sum_{k=1}^{\infty} \|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{k,r}+1} - x_{m_{k,r}}\|_{\alpha_r} < \infty
$$

for all $a_{1,r}, \ldots, a_{n-1,r} \in X$ and all $\alpha_r \in (0, 1)$. Now consider the diagonal sequence $m_k = m_{k,k}$. Let $r \in \mathbb{N}$. The sequence $\{m_k\}_{k=r}^{\infty}$ is a subsequence of $\{m_{k,r}\}_{k=r}^{\infty}$. Let $k \geq r$. There are pairs of integers $(u, v)$, $u < v$ such that $m_k = m_{u,r}$ and $m_{k+1} = m_{v,r}$. Then by the triangle inequality

$$
\|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{k+1}} - x_{m_k}\|_{\alpha_r} \leq \sum_{i=u}^{v-1} \|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{i+1,r}} - x_{m_{i,r}}\|_{\alpha_r}
$$

and therefore,

$$
\sum_{k=r}^{\infty} \|a_1, \ldots, a_{n-1}, x_{m_{k+1}} - x_{m_k}\|_\alpha \leq \sum_{j=r}^{\infty} \|a_1, \ldots, a_{n-1}, x_{m_{j+1,r}} - x_{m_{j,r}}\|_\alpha
$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$. The proof is complete. 

for all \(a_1, \ldots, a_{n-1} \in X\) and all \(\alpha \in (0, 1)\). The statement of the theorem follows. \(\square\)

The above theorem shows that many Cauchy sequence has a subsequence which has finite convergent. Therefore, it is natural to ask for an example of Cauchy sequence has a subsequence which has not finite convergent property.

**Example 4.2** We consider the set \(S\) consisting of all convergent real sequences. Let \(X\) be the space of all functions \(f: S \to \mathbb{R}\) equipped with the topology of pointwise convergence. This topology is generated by
\[
\|f_1, \ldots, f_{n-1, s}, f\|_{\alpha_s} = |f(s)|,
\]
for all \(f_1, \ldots, f_{n-1, s}, f \in X\) and all \(\alpha_s \in (0, 1)\), where \(s \in S\). Then consider the sequence \(f_n \in X\) defined by \(f_n(s) = s_n\) where \(s = (s_n) \in S\). The sequence \(f_n\) is a Cauchy sequence in \(X\) but there is no subsequence \(f_{n_k}\) such that
\[
\sum_{k=1}^{\infty} \|f_1, \ldots, f_{n-1, s}, f_{n_k+1} - f_{n_k}\|_{\alpha_s} < \infty
\]
for all \(s \in S\). We see this as follows. If \(n_1 < n_2 < n_3 < \ldots\) is a sequence then define \(s_n = (-1)^k \frac{1}{k}\) for \(k \leq n < n_{k+1}\). Then \(s = (s_n) \in S\) but
\[
\sum_{k=1}^{\infty} \|f_1, \ldots, f_{n-1, s}, f_{n_k+1} - f_{n_k}\|_{\alpha_s} = \sum_{k=1}^{\infty} |s_{n_k+1} - s_{n_k}| \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty.
\]

§5. Functions Preserving Convergence of Series in Fuzzy \(n\)-Normed Spaces

In this section we shall introduce the functions \(f: X \to X\) that preserve convergence of series in fuzzy \(n\)-normed spaces. Our work is an extension of functions \(f: \mathbb{R} \to \mathbb{R}\) that preserve convergence of series studied in [19] and [3].

We read in Cauchy’s condition in \((X, N)\) as follows: the series \(\sum_{k=1}^{\infty} x_k\) converges if and only if for every \(\epsilon > 0\) there is an \(N\) so that for all \(n \geq m \geq N\),
\[
\|a_1 \cdots, a_{n-1}, \sum_{k=m}^{n} x_k\| < \epsilon,
\]
where \(a_1 \cdots, a_{n-1} \in X\).

**Definition 5.1** A function \(f: X \times X \to X\) is said to be additive in fuzzy \(n\)-normed space \((X, N)\) if
\[
\|a_1, \cdots, a_{n-1}, f(x, y)\|_\alpha = \|a_1, \cdots, a_{n-1}, f(x)\|_\alpha + \|a_1, \cdots, a_{n-1}, f(y)\|_\alpha,
\]
for each \(x, y \in X, a_1, \cdots, a_{n-1} \in X\) and for all \(\alpha \in (0, 1)\).

**Definition 5.2** A function \(f: X \to X\) is convergence preserving (abbreviated CP) in \((X, N)\) if for every convergent series \(\sum_{k=1}^{\infty} x_k\), the series \(\sum_{k=1}^{\infty} f(x_k)\) is also convergent, i.e., for every \(a_1, \cdots, a_{n-1} \in X\),
\[
\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha < \infty
\]
whenever $\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, x_k\|_\alpha < \infty$.

**Theorem 5.1** Let $(X, N)$ be a fuzzy $n$-normed space and $f : X \to X$ be an additive and continuous function in the neighborhood $B(t, r, A)$. Then the function $f$ is CP of infinite series in $(X, N)$.

**Proof** Assume that $f$ is additive and continuous in $B(\alpha, \delta, A) = \{x \in X : \|a_1, \cdots, a_{n-1}, x\|_\alpha < \delta\}$, where $a_1, \cdots, a_{n-1} \in A$ and $\delta > 0$. From additivity of $f$ in $B(\alpha, \delta, A)$ implies that $f(0) = 0$. Let $\sum_{k=1}^{\infty} x_k$ be a absolute convergent series and $x_k \in X$ ($k = 1, 2, 3, \cdots$). We show that $\sum_{k=1}^{\infty} f(x_k)$ is also absolute convergent.

By Cauchy condition for convergence of series, there exists a $k \in \mathbb{N}$ such that for every $p \in \mathbb{N}$

$$\|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} x_j\|_\alpha < \frac{\delta}{2}.$$  

From this we have

$$\|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{\infty} x_j\|_\alpha < \frac{\delta}{2}.$$  

By the additivity of $f$ in $B(\alpha, \delta, A)$, we get

$$\|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{k+p} x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} f(x_j)\|_\alpha < \frac{\delta}{2}.$$  

Now, let $y_p = \sum_{j=k+1}^{k+p} x_j$ for $p = 1, 2, 3, \cdots$ and $y = \sum_{j=1}^{\infty} x_j$ belong to the neighborhood $B(\alpha, \delta, A)$. The function $f$ is continuous in $B(\alpha, \delta, A)$, i.e., $f(y_p) \to f(y)$ because $y_p \to y$ for $p \to \infty$. Hence

$$\lim_{p \to \infty} \|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{k+p} x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{\infty} x_j)\|_\alpha.$$  

This implies

$$\lim_{p \to \infty} \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} f(x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{\infty} f(x_j)\|_\alpha$$  

and this guarantee the convergence of the series $\sum_{j=k+1}^{\infty} f(x_j)$ and therefore the series $\sum_{j=1}^{\infty} f(x_j)$ must also be convergent in $X$, i.e., the function $f$ is CP infinite series in $(X, N)$.  

**References**


http://tatra.mat.savba.sk/Full/19/04DINDOS.ps