Euler-Savary’s Formula for the Planar Curves in
Two Dimensional Lightlike Cone

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Abstract: In this paper, we study the Euler-Savary’s formula for the planar curves in the lightlike cone. We first define the associated curve of a curve in the two dimensional lightlike cone $Q^2$. Then we give the relation between the curvatures of a base curve, a rolling curve and a roulette which lie on two dimensional lightlike cone $Q^2$.

Keywords: Lightlike cone, Euler Savary’s formula, Smarandache geometry, Smarandachely denied-free.


§1. Introduction

The Euler-Savary’s Theorem is well known theorem which is used in serious fields of study in engineering and mathematics.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. So the Euclidean geometry is just a Smarandachely denied-free geometry.

In the Euclidean plane $E^2$, let $c_B$ and $c_R$ be two curves and $P$ be a point relative to $c_R$. When $c_R$ roles without splitting along $c_B$, the locus of the point $P$ makes a curve $c_L$. The curves $c_B$, $c_R$ and $c_L$ are called the base curve, rolling curve and roulette, respectively. For instance, if $c_B$ is a straight line, $c_R$ is a quadratic curve and $P$ is a focus of $c_R$, then $c_L$ is the Delaunay curve that are used to study surfaces of revolution with the constant mean curvature, (see [1]). The relation between the curvatures of this curves is called as the Euler-Savary’s formula.

Many studies on Euler-Savary’s formula have been done by many mathematicians. For example, in [4], the author gave Euler-Savary’s formula in Minkowski plane. In [5], they expressed the Euler-Savary’s formula for the trajectory curves of the 1-parameter Lorentzian spherical motions.

On the other hand, there exists spacelike curves, timelike curves and lightlike(null) curves in semi-Riemannian manifolds. Geometry of null curves and its applications to general relativity in semi-Riemannian manifolds has been constructed, (see [2]). The set of all lightlike(null)

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vectors in semi-Riemannian manifold is called the lightlike cone. We know that it is important to study submanifolds of the lightlike cone because of the relations between the conformal transformation group and the Lorentzian group of the n-dimensional Minkowski space $E^3_n$ and the submanifolds of the n-dimensional Riemannian sphere $S^n$ and the submanifolds of the (n+1)-dimensional lightlike cone $Q^{n+1}$. For the studies on lightlike cone, we refer [3].

In this paper, we have done a study on Euler-Savary’s formula for the planar curves in two dimensional lightlike cone $Q^2$. However, to the best of author’s knowledge, Euler-Savary’s formula has not been presented in two dimensional lightlike cone $Q^2$. Thus, the study is proposed to serve such a need. Thus, we get a short contribution about Smarandache geometries.

This paper is organized as follows. In Section 2, the curves in the lightlike cone are reviewed. In Section 3, we define the associated curve that is the key concept to study the roulette, since the roulette is one of associated curves of the base curve. Finally, we give the Euler-Savary’s formula in two dimensional cone $Q^2$.

We hope that, these study will contribute to the study of space kinematics, mathematical physics and physical applications.

§ 2. Euler-Savary’s Formula in the Lightlike Cone $Q^2$

Let $E^3_1$ be the 3−dimensional Lorentzian space with the metric

$$g(x, y) = \langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in E^3_1$.

The lightlike cone $Q^2$ is defined by

$$Q^2 = \{ x \in E^3_1 : g(x, x) = 0 \}.$$

Let $x : I \to Q^2 \subset E^3_1$ be a curve, we have the following Frenet formulas (see [3])

$$x'(s) = \alpha(s)$$
$$\alpha'(s) = \kappa(s)x(s) - y(s)$$
$$y'(s) = -\kappa(s)\alpha(s),$$

where $s$ is an arclength parameter of the curve $x(s)$. $\kappa(s)$ is cone curvature function of the curve $x(s)$, and $x(s), y(s), \alpha(s)$ satisfy

$$\langle x, x \rangle = \langle y, y \rangle = \langle x, \alpha \rangle = \langle y, \alpha \rangle = 0,$$
$$\langle x, y \rangle = \langle \alpha, \alpha \rangle = 1.$$

For an arbitrary parameter $t$ of the curve $x(t)$, the cone curvature function $\kappa$ is given by

$$\kappa(t) = \frac{\langle \frac{dx}{dt}, \frac{d^2x}{dt^2} \rangle^2 - \langle \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2} \rangle \langle \frac{dx}{dt}, \frac{dx}{dt} \rangle}{2 \langle \frac{dx}{dt}, \frac{dx}{dt} \rangle^3}$$

(2.2)

Using an orthonormal frame on the curve $x(s)$ and denoting by $\pi, \tau, \beta$ and $\gamma$ the curvature, the torsion, the principal normal and the binormal of the curve $x(s)$ in $E^3_1$, respectively, we...
have
\[ x' = \alpha, \]
\[ \alpha' = \kappa x - y = \mp \beta, \]
where \( \kappa \neq 0, \langle \beta, \beta \rangle = \varepsilon = \pm 1, \langle \alpha, \beta \rangle = 0, \langle \alpha, \alpha \rangle = 1, \varepsilon \kappa < 0. \) Then we get
\[ \beta = \varepsilon \frac{\kappa x - y}{\sqrt{-2 \varepsilon \kappa}}, \quad \varepsilon \tau \gamma = \frac{\kappa'}{2 \sqrt{-2 \varepsilon \kappa}} (x + \frac{1}{\kappa} y). \tag{2.3} \]
Choosing
\[ \gamma = \sqrt{-\frac{\varepsilon \kappa}{2}} (x + \frac{1}{\kappa} y), \tag{2.4} \]
we obtain
\[ \tau = \mp \frac{1}{2} \left( \frac{\kappa'}{\kappa} \right). \tag{2.5} \]

**Theorem 2.1** The curve \( x : I \to Q^2 \) is a planar curve if and only if the cone curvature function \( \kappa \) of the curve \( x(s) \) is constant [3].

If the curve \( x : I \to Q^2 \subset E^3_1 \) is a planar curve, then we have following Frenet formulas

\[ x' = \alpha, \]
\[ \alpha' = \varepsilon \sqrt{-2 \varepsilon \kappa} \beta, \]
\[ \beta' = -\sqrt{-2 \varepsilon \kappa} \alpha. \tag{2.6} \]

**Definition 2.2** Let \( x : I \to Q^2 \subset E^3_1 \) be a curve with constant cone curvature \( \kappa \) (which means that \( x \) is a conic section) and arclength parameter \( s \). Then the curve
\[ x_A = x(s) + u_1(s) \alpha + u_2(s) \beta \tag{2.7} \]
is called the associated curve of \( x(s) \) in the \( Q^2 \), where \( \{ \alpha, \beta \} \) is the Frenet frame of the curve \( x(s) \) and \( \{ u_1(s), u_2(s) \} \) is a relative coordinate of \( x_A(s) \) with respect to \( \{ x(s), \alpha, \beta \} \).

Now we put
\[ \frac{dx_A}{ds} = \frac{\delta u_1}{ds} \alpha + \frac{\delta u_2}{ds} \beta. \tag{2.8} \]
Using the equation (2.2) and (2.6), we get
\[ \frac{dx_A}{ds} = \left( 1 + \frac{du_1}{ds} - \sqrt{-2 \varepsilon \kappa u_2} \right) \alpha + \left( u_1 \varepsilon \sqrt{-2 \varepsilon \kappa} + \frac{du_2}{ds} \right) \beta. \tag{2.9} \]
Considering the (2.8) and (2.9), we have
\[ \frac{\delta u_1}{ds} = \left( 1 + \frac{du_1}{ds} - \sqrt{-2 \varepsilon \kappa u_2} \right) \]
\[ \frac{\delta u_2}{ds} = \left( u_1 \varepsilon \sqrt{-2 \varepsilon \kappa} + \frac{du_2}{ds} \right) \tag{2.10} \]
Let $s_A$ be the arclength parameter of $x_A$. Then we write
\[ \frac{dx_A}{ds} = \frac{dx_A}{ds_A} \frac{ds_A}{ds} = v_1 \alpha + v_2 \beta \] (2.11)
and using (2.8) and (2.10), we get
\[ v_1 = 1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon \kappa u_2} \]
\[ v_2 = u_1 \varepsilon \sqrt{-2 \varepsilon \kappa} + \frac{du_2}{ds}. \] (2.12)

The Frenet formulas of the curve $x_A$ can be written as follows:
\[ \frac{d\alpha}{ds_A} = \varepsilon_A \sqrt{-2 \varepsilon \kappa A \beta A} \]
\[ \frac{d\beta}{ds_A} = -\sqrt{-2 \varepsilon \kappa A \alpha A}. \] (2.13)
where $\kappa_A$ is the cone curvature function of $x_A$ and $\varepsilon_A = \langle \beta_A, \beta_A \rangle = \pm 1$ and $\langle \alpha_A, \alpha_A \rangle = 1$.

Let $\theta$ and $\omega$ be the slope angles of $x$ and $x_A$ respectively. Then
\[ \kappa_A = \frac{d\omega}{ds_A} = (\pi + \frac{d\phi}{ds}) \frac{1}{\sqrt{v_1^2 + \varepsilon v_2^2}}, \] (2.14)
where $\phi = \omega - \theta$.

If $\beta$ is spacelike vector, then we can write
\[ \cos \phi = \frac{v_1}{\sqrt{v_1^2 + v_2^2}} \quad \text{and} \quad \sin \phi = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}. \]
Thus, we get
\[ \frac{d\phi}{ds} = \frac{d}{ds} (\cos \frac{1}{\sqrt{v_1^2 + v_2^2}}) \]
and (2.14) reduces to
\[ \kappa_A = (\pi + \frac{v_1 v_2 - v_1 v_2}{v_1^2 + v_2^2}) \frac{1}{\sqrt{v_1^2 + v_2^2}}. \]

If $\beta$ is timelike vector, then we can write
\[ \cosh \phi = \frac{v_1}{\sqrt{v_1^2 - v_2^2}} \quad \text{and} \quad \sinh \phi = \frac{v_2}{\sqrt{v_1^2 - v_2^2}} \]
and we get
\[ \frac{d\phi}{ds} = \frac{d}{ds} (\cosh^{-1} \frac{v_1}{\sqrt{v_1^2 - v_2^2}}). \]
Thus, we have
\[ \kappa_A = (\pi + \frac{v_1 v_2 - v_1 v_2}{v_1^2 - v_2^2}) \frac{1}{\sqrt{v_1^2 - v_2^2}}. \]

Let $x_B$ and $x_R$ be the base curve and rolling curve with constant cone curvature $\kappa_B$ and $\kappa_R$ in $Q^2$, respectively. Let $P$ be a point relative to $x_R$ and $x_L$ be the roulette of the locus of $P$. 

We can consider that \( x_L \) is an associated curve of \( x_B \) such that \( x_L \) is a planar curve in \( Q^2 \), then the relative coordinate \( \{w_1, w_2\} \) of \( x_L \) with respect to \( x_B \) satisfies
\[
\frac{\delta w_1}{ds_B} = 1 + \frac{dw_1}{ds_B} - \sqrt{-2\varepsilon_B^2 \kappa_B w_2}
\]
\[
\frac{\delta w_2}{ds_B} = w_1 \varepsilon_B \sqrt{-2\varepsilon_B^2 \kappa_B} + \frac{dw_2}{ds_B}
\]
(2.15)
by virtue of (2.10).

Since \( x_R \) roles without splitting along \( x_B \) at each point of contact, we can consider that \( \{w_1, w_2\} \) is a relative coordinate of \( x_L \) with respect to \( x_R \) for a suitable parameter \( s_R \). In this case, the associated curve is reduced to a point \( P \). Hence it follows that
\[
\frac{\delta w_1}{ds_R} = 1 + \frac{dw_1}{ds_R} - \sqrt{-2\varepsilon_R^2 \kappa_R w_2} = 0
\]
\[
\frac{\delta w_2}{ds_R} = w_1 \varepsilon_R \sqrt{-2\varepsilon_R^2 \kappa_R} + \frac{dw_2}{ds_R} = 0.
\]
(2.16)
Substituting these equations into (2.15), we get
\[
\frac{\delta w_1}{ds_B} = (\sqrt{-2\varepsilon_R^2 \kappa_R} - \sqrt{-2\varepsilon_B^2 \kappa_B})w_2
\]
\[
\frac{\delta w_2}{ds_B} = (\varepsilon_B \sqrt{-2\varepsilon_B^2 \kappa_B} - \varepsilon_R \sqrt{-2\varepsilon_R^2 \kappa_R})w_1.
\]
(2.17)
If we choose \( \varepsilon_B = \varepsilon_R = -1 \), then
\[
0 < (\frac{\delta w_1}{ds_B})^2 - (\frac{\delta w_2}{ds_B})^2 = (\sqrt{2 \kappa_R} - \sqrt{2 \kappa_B})^2(w_2^2 - w_1^2).
\]
(2.18)
Hence, we can put
\[
w_1 = r \sinh \phi, \quad w_2 = r \cosh \phi.
\]
Differentiating this equations, we get
\[
\frac{dw_1}{ds_R} = -r \sinh \phi + r \cosh \phi \frac{d\phi}{ds_R}
\]
\[
\frac{dw_2}{ds_R} = r \cosh \phi + r \sinh \phi \frac{d\phi}{ds_R}
\]
(2.19)
Providing that we use (2.16), then we have
\[
\frac{dw_1}{ds_R} = r \sqrt{2 \kappa_R} \cosh \phi - 1
\]
\[
\frac{dw_2}{ds_R} = r \sinh \phi \sqrt{2 \kappa_R}
\]
(2.20)
If we consider (2.19) and (2.20), then we get
\[
r \frac{d\phi}{ds_R} = -r \sqrt{2 \kappa_R} + \cosh \phi
\]
(2.21)
Therefore, substituting this equation into (2.14), we have
\[
r \pi_L = \pm 1 + \frac{\cosh \phi}{r \sqrt{2 \kappa_R - \sqrt{2 \kappa_B}}}
\]
(2.22)
If we choose $\varepsilon_B = \varepsilon_R = +1$, then from (2.17)

\[
0 < \left( \frac{\delta w_1}{ds_B} \right)^2 + \left( \frac{\delta w_2}{ds_B} \right)^2 = (\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B})^2 (w_1^2 + w_2^2) \tag{2.23}
\]

Hence we can put

\[
w_1 = r \sin \phi, \quad w_2 = r \cos \phi.
\]

Differentiating this equations, we get

\[
\frac{dw_1}{ds_R} = \frac{dr}{ds_R} \sin \phi + r \cos \phi \frac{d\phi}{ds_R} = r \sqrt{-2\kappa_R} \cos \phi - 1
\]

and

\[
\frac{dw_2}{ds_R} = \frac{dr}{ds_R} \cos \phi - r \sin \phi \frac{d\phi}{ds_R} = -r \sin \phi \sqrt{-2\kappa_R} \tag{2.24}
\]

Therefore, substituting this equation into (2.14), we have

\[
\frac{r}{ds_R} \kappa_L = \sqrt{-2\kappa_B} + \sqrt{-2\kappa_R} \left| \frac{\cos \phi}{r \sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}} \right|, \tag{2.26}
\]

where $\kappa_L = \sqrt{-2\varepsilon_L \kappa_L}$.

Thus we have the following Euler-Savary’s Theorem for the planar curves in two dimensional lightlike cone $Q^2$.

**Theorem 2.3** Let $x_R$ be a planar curve on the lightlike cone $Q^2$ such that it rolls without splitting along a curve $x_B$. Let $x_L$ be a locus of a point $P$ that is relative to $x_R$. Let $Q$ be a point on $x_L$ and $R$ a point of contact of $x_B$ and $x_R$ corresponds to $Q$ relative to the rolling relation. By $(r, \phi)$, we denote a polar coordinate of $Q$ with respect to the origin $R$ and the base line $x'_B | R$. Then curvatures $\kappa_B$, $\kappa_R$ and $\kappa_L$ of $x_B$, $x_R$ and $x_L$ respectively, satisfies

\[
\frac{r}{ds_R} \kappa_L = \pm 1 + \frac{\cosh \phi}{r \left| \sqrt{-2\kappa_R} - \sqrt{-2\kappa_B} \right|}, \quad \text{if} \quad \varepsilon_B = \varepsilon_R = -1,
\]

\[
\frac{r}{ds_R} \kappa_L = \frac{\sqrt{-2\kappa_B} + \sqrt{-2\kappa_R}}{\left| \sqrt{-2\kappa_R} - \sqrt{-2\kappa_B} \right|} \cdot \frac{\cos \phi}{r \left| \sqrt{-2\kappa_R} - \sqrt{-2\kappa_B} \right|}, \quad \text{if} \quad \varepsilon_B = \varepsilon_R = +1.
\]

**References**


