Lucas Gracefulness of Almost and Nearly for Some Graphs

M.A. Perumal, S. Navaneethakrishnan and A. Nagarajan

Department of Mathematics, National Engineering College,
K.R. Nagar, Kovilpatti, Tamil Nadu, India

E-mail: meetperumal.ma@gmail.com, snk.voc@gmail.com, nagarajan.voc@gmail.com

Abstract: Let $G$ be a $(p, q)$ - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_q\}$, $(a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If $G$ admits Lucas graceful labeling, then $G$ is said to be Lucas graceful graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_{a-1}, l_a+1\}$, $(a \in N)$, is said to be almost Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ or $\{l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then $G$ is called almost Lucas graceful graph if it admits almost Lucas graceful labeling. Also, an injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_u\}$, $(a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u,v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{a-1}, l_{a+1}, l_{a+2}, \cdots, l_{b, b \leq a}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If $G$ admits nearly Lucas graceful labeling, then $G$ is said to be nearly Lucas graceful graph. In this paper, we show that the graphs $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n}@P_t$ and $C_n$ are nearly Lucas graceful graphs.

Key Words: Smarandache-Fibonacci triple, super Smarandache-Fibonacci graceful graph, graceful labeling, Lucas graceful labeling, almost Lucas graceful labeling and nearly Lucas graceful labeling.

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§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A cycle of length $n$ is denoted by $C_n \cdot G^+$ is a graph obtained from the graph $G$ by attaching pendant vertex to each vertex of $G$. The concept of graceful labeling was introduced by Rosa [3] in 1967. A function $f$ is called a graceful labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{1, 2, 3, \cdots, q\}$ such that when each edge $uv$ is assigned the label

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\[ |f(u) - f(v)|, \text{ the resulting edge labels are distinct.} \]
The notion of Fibonacci graceful labeling was introduced by K.M.Kathiresan and S.Amutha [4]. We call a function \( f \), a Fibonacci graceful label labeling of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the set \( \{0, 1, 2, \ldots, F_q\} \), where \( F_q \) is the \( q \)th Fibonacci number of the Fibonacci series \( F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots \) and each edge \( uv \) is assigned the label \( |f(u) - f(v)| \). Based on the above concept we define the following.

A Smarandache-Fibonacci triple is a sequence \( S(n) \), \( n \geq 0 \) such that \( S(n) = S(n-1) + S(n-2) \), where \( S(n) \) is the Smarandache function for integers \( n \geq 0 \). Clearly, it is a generalization of Fibonacci sequence and Lucas sequence. Let \( G \) be a \((p, q)\)-graph and \( \{S(n)|n \geq 0\} \) a Smarandache-Fibonacci triple. An bijection \( f : V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\} \) is said to be a super Smarandache-Fibonacci graceful graph if the induced edge labeling \( f^{*}(uv) = |f(u) - f(v)| \) is a bijection onto the set \( \{S(1), S(2), \ldots, S(q)\} \). Particularly, if \( S(n), n \geq 0 \) is just the Lucas sequence, such a labeling \( f : V(G) \to \{l_0, l_1, l_2, \ldots, l_q\} \) \((a \in N)\) is said to be Lucas graceful labeling if the induced edge labeling \( f_1(uv) = |f(u) - f(v)| \) is a bijection on to the set \( \{l_1, l_2, \ldots, l_q\} \). If \( G \) admits Lucas graceful labeling, then \( G \) is said to be Lucas graceful graph. An injective function \( f : V(G) \to \{l_0, l_1, l_2, \ldots, l_{a-1}, l_{a+1}\} \) \((a \in N)\), is said to be almost Lucas graceful labeling if the induced edge labeling \( f_1(uv) = |f(u) - f(v)| \) is a bijection onto the set \( \{l_1, l_2, \ldots, l_q\} \) or \( \{l_1, l_2, \ldots, l_{q-1}, l_{q+1}\} \) with the assumption of \( l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, \ldots \). Then \( G \) is called almost Lucas graceful graph if it admits almost Lucas graceful labeling. Also, an injective function \( f : V(G) \to \{l_0, l_1, l_2, \ldots, l_a\} \) \((a \in N)\), is said to be nearly Lucas graceful labeling if the induced edge labeling \( f_1(uv) = |f(u) - f(v)| \) onto the set \( \{l_1, l_2, \ldots, l_{i-1}, l_{i+1}, l_{i+2}, \ldots, l_{j-1}, l_{j+1}, l_{j+2}, \ldots, l_{k-1}, l_{k+1}, l_{k+2}, \ldots, l_b\} \) \((b \in N \text{ and } b \leq a)\) with the assumption of \( l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, \ldots \). If \( G \) admits nearly Lucas graceful labeling, then \( G \) is said to be nearly Lucas graceful graph. In this paper, we show that the graphs \( S_m, S_n@P_t \) and \( F_m@P_n \) are almost Lucas graceful graphs. Also we show that the graphs \( S_m@P_t \) and \( C_n \) are nearly Lucas graceful graphs.

\section*{2. Almost Lucas Graceful Graphs}

In this section, we show that some graphs namely \( S_m, S_n@P_t \) and \( F_m@P_n \) are almost Lucas graceful graphs.

**Definition 2.1** Let \( G \) be a \((p, q)\) - graph. An injective function \( f : V(G) \to \{l_0, l_1, l_2, \ldots, l_{a-1}, l_{a+1}\} \), \( a \in N \), is said to be almost Lucas graceful labeling if the induced edge labeling \( f_1(uv) = |f(u) - f(v)| \) is a bijection onto the set \( \{l_1, l_2, \ldots, l_q\} \) or \( \{l_1, l_2, \ldots, l_{q-1}, l_{q+1}\} \). Then \( G \) is called almost Lucas graceful graph if it admits almost Lucas graceful labeling.

**Definition 2.2** ([2]) \( S_m, n \) denotes a star with \( n \) spokes in which each spoke is a path of length \( m \).

**Theorem 2.3** \( S_m, n \) is an almost Lucas graceful graph when \( m \equiv 1(\mod 2) \) and \( n \equiv 0(\mod 3) \)

**Proof** Let \( G = S_m, n \). Let \( V(G) = \{u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \) be the vertex set of
Let $E(G) = \{u_0 u_{i_1} : 1 \leq i \leq m\} \cup \{u_{i,j} u_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n - 1\}$ be the edge set of $G$. So, $|V(G)| = mn + 1$ and $|E(G)| = mn$. Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \ldots, l_a\}, a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \ldots, m - 2$ and $j \equiv 1 (\bmod 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \ldots, m - 1$ and $j \equiv 0 (\bmod 2)$, $f(u_{i,j}) = l_{n+2-2j}, 1 \leq j \leq n$. For $s = 1, 2, \ldots, \frac{n-3}{3}$, $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s}, 3s-2 \leq j \leq 3s$, and for $s = \frac{n}{3}$, $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s}, 3s-2 \leq j \leq 3s-1$. We claim that the edge labels are distinct. Let

$$E_1 = \bigcup_{i \equiv 1 (\bmod 2)}^{m} \{f_1(u_0 u_{i_1})\} = \bigcup_{i \equiv 1 (\bmod 2)}^{m} \{|f(u_0) - f(u_{i_1})|\}$$

$$= \bigcup_{i \equiv 1 (\bmod 2)}^{m} \{|l_0 - l_{n(i-1)+1}|\} = \bigcup_{i \equiv 1 (\bmod 2)}^{m} \{l_{n(i-1)+1}\}$$

$$= \{l_1, l_{2n+1}, l_{4n+1}, \ldots, l_{(m-1)n+1}\},$$

$$E_2 = \bigcup_{i \equiv 0 (\bmod 2)}^{m} \{f_1(u_0 u_{i_1})\} = \bigcup_{i \equiv 0 (\bmod 2)}^{m} \{|f(u_0) - f(u_{i_1})|\}$$

$$= \bigcup_{i \equiv 0 (\bmod 2)}^{m} \{|l_0 - l_{ni}|\} = \bigcup_{i \equiv 0 (\bmod 2)}^{m} \{l_{ni}\}$$

$$= \{l_{2n}, l_{4n}, \ldots, l_{(m-1)n}\},$$

$$E_3 = \bigcup_{i \equiv 1 (\bmod 2)}^{m-2} \bigcup_{j \equiv 1 (\bmod 2)}^{n-1} \{f_1(u_{i,j} u_{i,j+1})\} = \bigcup_{i \equiv 1 (\bmod 2)}^{m-2} \bigcup_{j \equiv 1 (\bmod 2)}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\}$$

$$= \bigcup_{i \equiv 1 (\bmod 2)}^{m-2} \bigcup_{j \equiv 1 (\bmod 2)}^{n-1} \{|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}|\} = \bigcup_{i \equiv 1 (\bmod 2)}^{m-2} \bigcup_{j \equiv 1 (\bmod 2)}^{n-1} \{l_{n(i-1)+2j}\}$$

$$= \bigcup_{i \equiv 1 (\bmod 2)}^{m-2} \{l_{n(i-1)+2}, l_{n(i-1)+4}, \ldots, l_{n(i-1)+2n-2}\}$$

$$= \{l_2, l_4, \ldots, l_{2n-2}\} \cup \{l_{2n+2}, l_{2n+4}, \ldots, l_{4n-2}\} \cup \ldots \cup \{l_{(m-3)n+2}, l_{(m-3)n+4}, \ldots, l_{(m-3)n+2n-2}\}$$

$$= \{l_2, l_4, \ldots, l_{2n-2}, l_{2n+2}, \ldots, l_{4n-2}, \ldots, l_{(m-3)n+2}, \ldots, l_{mn-n-2}\},$$
\[ E_4 = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{ f_1(u_{i,j}u_{i,j+1}) \} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{ |f(u_{i,j}) - f(u_{i,j+1})| \} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{ |l_{ni-2j+2} - l_{ni-2j}| \} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{ l_{ni-2j+1} \} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{ l_{ni-1}, l_{ni-3}, \ldots, l_{ni-2n+3} \} = \{ l_{2n-1}, l_{2n-3}, \ldots, l_3 \} \bigcup \{ l_{4n-1}, l_{4n-3}, \ldots, l_{2n+3} \} \bigcup \cdots \bigcup \{ l_{(m-1)n-1}, l_{(m-1)n-3}, \ldots, l_{mn-3n+3} \} = \{ l_{2n-1}, l_{2n-3}, \ldots, l_3, l_{4n-1}, l_{4n-3}, \ldots, l_{2n+3}, \ldots, l_{(m-1)n-1}, l_{(m-1)n-3}, \ldots, l_{mn-3n+3} \}, \]

\[ E_5 = \bigcup_{s=1}^{n-3} \{ f_1(u_{m,j}u_{m,j+1}) : 3s - 2 \leq j \leq 3s - 1 \} = \bigcup_{s=1}^{n-3} \{ |f(u_{m,j}) - f(u_{m,j+1})| : 3s - 2 \leq j \leq 3s - 1 \} = \bigcup_{s=1}^{n-3} \{ |l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4}| : 3s - 2 \leq j \leq 3s - 1 \} = \bigcup_{s=1}^{n-3} \{ l_{n(m-1)+2j-3s+3} : 3s - 2 \leq j \leq 3s - 1 \} = \{ l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+6}, l_{n(m-1)+7} \} \bigcup \cdots \bigcup \{ l_{n(m-1)+2n-10-n+3+3}, l_{n(m-1)+2n-8-n+3+3} \} = \{ l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+6}, l_{n(m-1)+7}, \ldots, l_{n(m-1)+n-4}, l_{n(m-1)+n-2} \} = \{ l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+6}, l_{n(m-1)+7}, \ldots, l_{mn-4}, l_{mn-2} \}. \]

We find the edge labeling between the end vertex of \( s^{th} \) loop and the starting vertex of \((s+1)^{th}\) loop and \( s = 1, 2, \ldots, \frac{n-3}{3} \). Let

\[ E_6 = \bigcup_{s=1}^{n-3} \{ f_1(u_{m,j}u_{m,j+1}) : j = 3s \} = \bigcup_{s=1}^{n-3} \{ |f(u_{m,j}) - f(u_{m,j+1})| : j = 3s \} = \{ |f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \ldots, |f(u_{m,n-3}) - f(u_{m,n-2})| \} = \{ l_{(m-1)n+8-3} - l_{(m-1)n+10-6}, l_{(m-1)n+14-6} - l_{(m-1)n+16-9}, \ldots, l_{(m-1)n+2n-4-n+3} - l_{(m-1)n+2n-2-n} \} = \{ l_{(m-1)n+5} - l_{(m-1)n+4}, l_{(m-1)n+8} - l_{(m-1)n+7}, \ldots, l_{(m-1)n+n-1} - l_{(m-1)n+n-2} \} = \{ l_{(m-1)n+3}, l_{(m-1)n+6}, \ldots, l_{mn-3} \}. \]
For $s = \frac{n}{3}$, let

$$E_7 = \{f_1(u_{m,j}u_{m,j+1}) : 3s - 2 \leq j \leq 3s - 1\} = \{f(u_{m,j}) - f(u_{m,j+1}) : 3s - 2 \leq j \leq 3s - 1\}$$

$$= \{|l_{(m-1)n+2n-2-n} - l_{(m-1)n+2n-2-n}|, l_{(m-1)n+2n-2-n} - l_{(m-1)n+2n-2-n}\}$$

$$= \{|l_{(m-1)n+n-2} - l_{(m-1)n+n}|, l_{(m-1)n+n} - l_{(m-1)n+n+2}\}$$

$$= \{l_{(m-1)n+n-1}, l_{(m-1)n+n+1}\} = \{l_{mn-1}, l_{mn+1}\}.$$ 

Now, $E = \sum_{i=1}^{7} E_i = \{l_1, l_2, ..., l_{mn-1}, l_{mn+1}\}$. So, the edge labels of $G$ are distinct. Therefore, $f$ is an almost Lucas graceful labeling. Thus $G = S_{m,n}$ is an almost Lucas graceful graph, when $m \equiv 1(\text{mod } 2)$ and $n \equiv 0(\text{mod } 3)$.

Example 2.4 An almost Lucas graceful labeling of $S_{7,9}$ is shown in Fig.2.1.

**Fig.2.1** $S_{7,9}$

Definition 2.5([2]) The graph $G = S_{m,n}@P_t$ consists of $S_{m,n}$ and a path $P_t$ of length $t$ which is attached with the maximum degree of the vertex of $S_{m,n}$.

Theorem 2.6 $S_{m,n}@P_t$ is an almost Lucas graceful graph when $m \equiv 0(\text{mod } 2)$ and $t \equiv 0(\text{mod } 3)$.

Proof Let $G = S_{m,n}@P_t$ with $m \equiv 0(\text{mod } 3)$ and $t \equiv 0(\text{mod } 3)$. Let

$$V(G) = \{u_0, u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{v_k : 1 \leq k \leq t\},$$

$$E(G) = \{u_0u_{i,1} : 1 \leq i \leq m\} \cup \{u_{i,j}u_{ij+1} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1\}$$

$$\cup \{u_0v_1\} \cup \{v_kv_{k+1} : 1 \leq k \leq t - 1\}.$$
be the vertex set and edge set of $G$, respectively. Thus $|V(G)| = mn + t + 1$ and $|E(G)| = mn + t$. Define $f : V(G) \to \{l_0, l_1, l_2, \ldots, l_a\}, a \in \mathbb{N}$ by $f(u_0) = l_0$. For $i = 1, 2, \ldots, m$ and for $i \equiv 1(\text{mod } 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \ldots, m$ and for $i \equiv 1(\text{mod } 2)$, $f(u_{i,j}) = l_{ni-2j+2}, 1 \leq j \leq n$. For $s = 1, 2, \ldots, \frac{t-3}{3}$, $f(v_k) = l_{mn+2k-3s+2}$, $3s - 2 \leq k \leq 3s$ and for $s = \frac{t}{3}$, $f(v_k) = l_{mn+2k-3s+2}$, $3s - 2 \leq k \leq 3s - 1$. We claim that the edge labels are distinct. Let

$$E_1 = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{f_1(u_0 u_{i,1})\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{|f(u_0) - f(u_{i,1})|\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{|l_0 - l_{n(i-1)+1}|\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{l_{n(i-1)+1}\} = \{l_1, l_{2n+1}, l_{4n+1}, \ldots, l_{n(m-1)+1}\},$$

$$E_2 = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{f_1(u_0 u_{i,1})\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{|f(u_0) - f(u_{i,1})|\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{|l_0 - l_{n1}|\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \{l_{n1}\} = \{l_{2n}, l_{4n}, \ldots, l_{mn}\},$$

$$E_3 = \bigcup_{i \equiv 1(\text{mod } 2)}^m \bigcup_{j=1}^{n-1} \{f_1(u_{i,j} u_{i,j+1})\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \bigcup_{j=1}^{n-1} \{|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}|\} = \bigcup_{i \equiv 1(\text{mod } 2)}^m \bigcup_{j=1}^{n-1} \{l_{n(i-1)+2j}\} = \{l_{n(i-1)+2}, l_{n(i-1)+4}, \ldots, l_{n(i-1)+2n-2}\} = \{l_2, l_4, \ldots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \ldots, l_{4n-2}\} \bigcup \cdots \bigcup \{l_{n(m-2)+2}, l_{n(m-2)+4}, \ldots, l_{mn-2}\} = \{l_2, l_4, \ldots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \ldots, l_{4n-2}, \ldots, l_{n(m-2)+2}, l_{n(m-2)+4}, \ldots, l_{mn-2}\}.
\[ E_4 = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n-1} \{ f_1(u_{ij}u_{i,j+1}) \} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n-1} \{ |f(u_{ij}) - f(u_{i,j+1})| \} \]

\[ = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n-1} \{ |l_{ni-2j+2} - l_{ni-2j}| \} \]

\[ = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n-1} \{ l_{ni-2j+1} \} = \bigcup_{i=1}^{m} \{ l_{ni-1}, l_{ni-3}, \ldots, l_{ni-2n+3} \} \]

\[ = \{ l_{2n-1}, l_{2n-3}, \ldots, l_3 \} \cup \{ l_{4n-1}, l_{4n-3}, \ldots, l_{2n+3} \} \cup \ldots \cup \{ l_{mn-1}, l_{mn-3}, \ldots, l_{mn-2n+3} \} \]

\[ = \{ l_{2n-1}, l_{2n-3}, \ldots, l_3, l_{4n-1}, l_{4n-3}, \ldots, l_{2n+3}, \ldots, l_{mn-1}, l_{mn-3}, \ldots, l_{mn-2n+3} \} \]

\[ E'_1 = \{ f_1(u_0v_1) \} = \{ |f(u_0) - f(v_1)| \} = \{ |l_0 - l_{mn+1}| \} = \{ l_{mn+1} \} , \]

\[ E'_2 = \bigcup_{s=1}^{t/3} \{ f_1(v_kv_{k+1}) : 3s - 2 \leq k \leq 3s - 1 \} \]

\[ = \bigcup_{s=1}^{t/3} \{ |f(v_k) - f(v_{k+1}) : 3s - 2 \leq k \leq 3s - 1| \} \]

\[ = \bigcup_{s=1}^{t/3} \{ |l_{mn+2k+2-3s} - l_{mn+2k+4-3s}| : 3s - 2 \leq k \leq 3s - 1 \} \]

\[ = \bigcup_{s=1}^{t/3} \{ l_{mn+2k+3-3s} : 3s - 2 \leq k \leq 3s - 1 \} \]

\[ = \{ l_{mn+2}, l_{mn+4} \} \cup \{ l_{mn+5}, l_{mn+7} \} \cup \ldots \cup \{ l_{mn+t-4}, l_{mn+t-2} \} \]

\[ = \{ l_{mn+2}, l_{mn+4}, l_{mn+5}, l_{mn+7}, \ldots, l_{mn+t-4}, l_{mn+t-2} \} \]

We find the edge labeling between the end vertex of \( s^{th} \) loop and the starting vertex of \((s+1)^{th}\) loop for integers \( s = 1, 2, \ldots, \frac{t-3}{3} \). Let

\[ E'_3 = \bigcup_{s=1}^{t/3} \{ f_1(u_{3s}u_{3s+1}) \} \]

\[ = \{ |f(u_{3s}) - f(u_{3s+1})| \} \]

\[ = \{ |f(u_3) - f(u_4)|, |f(u_6) - f(u_7)|, \ldots, |f(t-3) - f(t-2)| \} \]

\[ = \{ |l_{mn+8-3} - l_{mn+10-6}|, |l_{mn+14-6} - l_{mn+16-9}|, \ldots, |l_{mn+2t-4-t+3} - l_{mn+2t-2-t}| \} \]

\[ = \{ l_{mn+5} - l_{mn+4}, l_{mn+8} - l_{mn+7}, \ldots, l_{mn+t-1} - l_{mn+t-2} \} \]

\[ = \{ l_{mn+3}, l_{mn+6}, \ldots, l_{mn+t-3} \} . \]
For $s = \frac{t}{3}$, let

\[
E'_4 = \{f_1(v_kv_{k+1}) : 3s - 2 \leq k \leq 3s - 1\}
\]

\[
= \{|f(v_k) - f(v_{k+1})| : 3s - 2 \leq k \leq 3s - 1\}
\]

\[
= \{|l_{mn+2t-4+2-t} - l_{mn+2t-2+2-t} - l_{mn+2t-2-2-t} - l_{mn+2t+2-t}|\}
\]

\[
= \{|l_{mn+t-2} - l_{mn+t+2} - l_{mn+t+1}|\}
\]

Now, $E = \bigcup_{i=1}^{4}(E_i \cup E'_i) = \{l_1, l_2, \ldots, l_{mn}, \ldots, l_{mn+t-1}, l_{mn+t+1}\}$. So, the edge labels of $G$ are distinct. Therefore, $f$ is an almost Lucas graceful graph. Thus $G = S_{m,n}@P_t$ is an almost Lucas graceful graph when $m \equiv 0$(mod 2) and $t \equiv 0$(mod 3).

**Example 2.7** An almost Lucas graceful labeling on $S_{4,7}@P_6$ is shown in Fig.2.2.

![Diagram](image.png)

**Fig.2.2** $S_{4,7}@P_6$

**Definition 2.8**(2) The graph $G = F_m@P_n$ consists of a fan $F_m$ and a path $P_n$ of length $n$ which is attached with the maximum degree of the vertex of $F_m$.

**Theorem 2.9** $F_m@P_n$ is almost Lucas graceful graph when $n \equiv 0$(mod 3).

**Proof** Let $v_1, v_2, \cdots, v_{m+1}$ and $u_0$ be the vertices of a Fan $F_m$. Let $u_1, u_2, \cdots, u_n$ be the vertices of a path $P_n$. Let $G = F_m@P_n$, $|V(G)| = m + n + 2$ and $|E(G)| = 2m + n + 1$. Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_{q+2}\}$ by $f(u_0) = l_0$: $f(v_i) = l_{2i-1}$: $f(u_j) = l_{2m+2j-3s+3}$, $3s - 2 \leq j \leq 3s$. We claim that the edge labels are distinct. Let

\[
E_1 = \bigcup_{i=1}^{m} \{f_1(v_i,v_{i+1})\} = \bigcup_{i=1}^{m} \{|f(v_i) - f(v_{i+1})|\}
\]

\[
= \bigcup_{i=1}^{m} \{|l_{2i-1} - l_{2i+1}|\}
\]

\[
= \bigcup_{i=1}^{m} \{l_{2i}\} = \{l_2, l_4, \ldots, l_{2m}\},
\]
We find the edge labeling between the end vertex of $s$th loop for $G$. Thus the edge labels of $G$ are distinct. Therefore, $f$ is an almost Lucas graceful labeling.

We find the edge labeling between the end vertex of $s$th loop and the starting vertex of $(s+1)^{th}$ loop for $s = 1, 2, \cdots, \frac{n}{3} - 1$. Let

$$E_5 = \bigcup_{s=1}^{\frac{n}{3} - 1} \{f_1(u_ju_{j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n}{3} - 1} \{|f(u_j) - f(u_{j+1})| : j = 3s\}$$

$$= \{l_{2m+6+3-3} - l_{2m+8+3-6}, |l_{2m+12+3-6} - l_{2m+14+3-9}|,$$

$$\cdots, |l_{2m+2n-6+3-n+3} - l_{2m+2n+4+3-n}|\}$$

$$= \{l_{2m+6} - l_{2m+5}, |l_{2m+9} - l_{2m+8}|, |l_{2m+n} - l_{2m+n-1}|\}$$

$$= \{l_{2m+4}, l_{2m+7}, \cdots, l_{2m+n-2}\}.$$ 

For $s = \frac{n}{3}$, let

$$E_6 = \{f_1(u_ju_{j+1}) : 3s - 2 \leq j \leq 3s - 1\}$$

$$= \{|f(u_j) - f(u_{j+1})| : 3s - 2 \leq j \leq 3s - 1\}$$

$$= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\}$$

$$= \{|l_{2m+2n-4+3-n} - l_{2m+2n-2+3-n}, |l_{2m+2n-2+3-n} - l_{2m+2n+3-n}|\}$$

$$= \{|l_{2m+n-1} - l_{2m+n+1}, |l_{2m+n+1} - l_{2m+n+3}|\}$$

$$= \{l_{2m+n}, l_{2m+n+2}\}.$$ 

Now, $E = \bigcup_{i=1}^{6} E_i = \{l_1, l_2, \cdots, l_{2m+1}, l_{2m+2}, \cdots, l_{2m+n-2}, l_{2m+n-1}, l_{2m+n}, l_{2m+n+2}\}$. So, the edge labels of $G$ are distinct. Therefore, $f$ is an almost Lucas graceful labeling.

Thus $G = F_m \circ P_n$ is an almost Lucas graceful graph when $n \equiv 0(\text{mod } 3).$
Example 2.10 An almost Lucas graceful labeling on $F_5 @ P_6$ is shown in Fig. 2.3.

§3. Nearly Lucas Graceful Graphs

In this section, we show that the graphs $S_{m,n} @ P_t$ and $C_n$ are nearly Lucas graceful graphs.

Definition 3.1 Let $G$ be a $(p,q)$-graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \ldots, l_n\}$, $(a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u,v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \ldots, l_{i-1}, l_{i+1}, l_{i+2}, \ldots, l_{j-1}, l_{j+1}, l_{j+2}, \ldots, l_{k-1}, l_{k+1}, l_{k+2}, \ldots, l_b \}$ ($b \leq a$) with the assumption of $l_0 = 0$, $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, $l_4 = 7$, $l_5 = 11$, etc. If $G$ admits nearly Lucas graceful labeling, then $G$ is said to be nearly Lucas graceful graph.

Theorem 3.2 $S_{m,n} @ P_t$ is a nearly Lucas graceful graph when $n \equiv 1, 2 (mod \ 3)$, $m \equiv 1 (mod \ 2)$ and $t = 1, 2 (mod \ 3)$.

Proof Let $G = S_{m,n} @ P_t$ with $V(G) = \{u_0, u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{v_k : 1 \leq k \leq t\}$. Let $E(G) = \{u_0 u_{i,j} : 1 \leq i \leq m\} \cup \{u_{i,j} u_{i,j+1} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{u_0 v_1\} \cup \{v_{k} v_{k+1} : 1 \leq k \leq t - 1\}$ be the edge set of $G$. So, $|V(G)| = mn + t + 1$ and $|E(G)| = mn + t$.

Define $f : V(G) \rightarrow \{l_0, l_1, \ldots, l_t\}$, $a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \ldots, m$ and for $i \equiv 1 (mod \ 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}$, $1 \leq j \leq n$. For $i = 1, 2, \ldots, m$ and for $i \equiv 0 (mod \ 2)$, $f(u_{i,j}) = l_{n(i-2)+2j+2}$, $1 \leq j \leq n$. For $s = 1, 2, \ldots, \frac{n-2}{3} - 1$ or $s = 1, 2, \ldots, \frac{n-1}{3} - 1$ or $s = 1, 2, \ldots, \frac{n}{3} - 1$, $f(u_{m,j}) = l_{mn+2(j+1)-3s}$, $3s - 2 \leq j \leq 3s$. For $s = \frac{n-2}{3}$ or $s = \frac{n-1}{3}$ or $s = \frac{n}{3}$, $f(u_{m,j}) = l_{mn+2(j+1)-3s}$, $3s - 2 \leq j \leq 3s - 1$. For $r = 1, 2, \ldots, \frac{t-2}{3}$ or $r = 1, 2, \ldots, \frac{t-1}{3}$ or $r = 1, 2, \ldots, \frac{t}{3}$, $f(v_k) = l_{mn+2k+3-3r}$, $3r - 2 \leq j \leq 3r - 1$. We claim that the edge labels
are distinct. Let

\[
E_1 = \bigcup_{i \equiv 1 \pmod{2}}^m \{f_1(u_0u_{i,1})\} = \bigcup_{i \equiv 1 \pmod{2}}^m \{|f(u_0) - f(u_{i,j})|\}
\]

\[
= \bigcup_{i \equiv 1 \pmod{2}} \{|l_0 - l_{(i-1)n+1}|\} = \bigcup_{i \equiv 1 \pmod{2}}^m \{l_{(i-1)n+1} = \{l_1, l_{2n+1}, \ldots, l_{(m-1)n+1}\},
\]

\[
E_2 = \bigcup_{i \equiv 0 \pmod{2}}^m \{f_1(u_0u_{i,1})\} = \bigcup_{i \equiv 0 \pmod{2}}^m \{|f(u_0) - f(u_{i,1})|\}
\]

\[
= \bigcup_{i \equiv 0 \pmod{2}}^m \{l_0 - l_{1n}\} = \bigcup_{i \equiv 0 \pmod{2}}^m \{l_{1n} = \{l_{2n}, l_{4n}, \ldots, l_{(m-1)n}\},
\]

\[
E_3 = \bigcup_{i \equiv 1 \pmod{2}}^{m-2} \bigcup_{j = 1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{i \equiv 1 \pmod{2}}^{m-2} \bigcup_{j = 1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\}
\]

\[
= \bigcup_{i \equiv 1 \pmod{2}}^{m-2} \bigcup_{j = 1}^{n-1} \{|l_{(i-1)n+2j-1} - l_{(i-1)n+2j+1}|\} = \bigcup_{i \equiv 1 \pmod{2}}^{m-2} \bigcup_{j = 1}^{n-1} \{l_{(i-1)n+2j}\}
\]

\[
= \bigcup_{i \equiv 1 \pmod{2}}^{m-2} \{l_{(i-1)n+2j}, l_{(i-1)n+4j}, \ldots, l_{(i-1)n+2n-2}\}
\]

\[
= \{l_2, l_4, \ldots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \ldots, l_{4n-2}\} \bigcup \ldots \bigcup \{l_{(m-3)n+2}, l_{(m-3)n+4}, \ldots, l_{mn-2}\}
\]

\[
= \{l_2, l_4, \ldots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \ldots, l_{4n-2}, \ldots, l_{(m-3)n+2}, l_{(m-3)n+4}, \ldots, l_{mn-2}\},
\]

\[
E_4 = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j = 1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j = 1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\}
\]

\[
= \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j = 1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j = 1}^{n-1} \{l_{ni-2j+1}\}
\]

\[
= \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \{l_{in-1}, l_{in-3}, \ldots, l_{in-2n+3}\}
\]

\[
= \{l_{2n-1}, l_{2n-3}, \ldots, l_3\} \bigcup \{l_{4n-1}, l_{4n-3}, \ldots, l_{2n+3}\} \bigcup \ldots \bigcup \{l_{(m-1)n-1}, l_{(m-1)n-3}, \ldots, l_{mn-3n+3}\}
\]

\[
= \{l_{2n-1}, l_{2n-3}, \ldots, l_{4n-1}, l_{4n-3}, \ldots, l_{2n+3}, \ldots, l_{(m-1)n-1}, l_{(m-1)n-3}, \ldots, l_{mn-3n+3}\}.
\]
For $n \equiv 1 (\text{mod } 3)$ and $s = 1, 2, \cdots, \frac{n-4}{3}$, let

$$E_5 = \bigcup_{s=1}^{\frac{n-4}{3}} \{ f_1(u_{m,j}u_{m,j+1}) : 3s - 2 \leq j \leq 3s - 1 \}$$

$$= \bigcup_{s=1}^{\frac{n-4}{3}} \{ |f(u_{m,j}) - f(u_{m,j+1})| : 3s - 2 \leq j \leq 3s - 1 \}$$

$$= \bigcup_{s=1}^{\frac{n-4}{3}} \{ |l(m-1)n+2j-3s+2 - l(m-1)n+2j-3s+4| : 3s - 2 \leq j \leq 3s - 1 \}$$

$$= \bigcup_{s=1}^{\frac{n-4}{3}} \{ l(m-1)n+2j-3s+3 : 3s - 2 \leq j \leq 3s - 1 \}$$

$$= \{ l(m-1)n+2, l(m-1)n+4 \} \cup \{ l(m-1)n+5, l(m-1)n+7 \} \cup \cdots \cup \{ l(m-1)n+n-4, l(m-1)n+n-2 \}$$

$$= \{ l(m-1)n+2, l(m-1)n+4, l(m-1)n+5, l(m-1)n+7, \cdots, l(mn-4), l_{mn-2} \}.$$

We find the edge labeling between the end vertex of $s^{th}$ loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \cdots, \frac{n-4}{3}$. Let

$$E_6 = \bigcup_{s=1}^{\frac{n-1}{3}} \{ f_1(u_{m,j}u_{m,j+1}) : j = 3s \} = \bigcup_{s=1}^{\frac{n-1}{3}} \{ |f(u_{m,j}) - f(u_{m,j+1})| : j = 3s \}$$

$$= \{ |f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \cdots, |f(u_{m,n-1}) - f(u_{m,n})| \}$$

$$= \{ |l(m-1)n+5 - l(m-1)n+4|, |l(m-1)n+7|, \cdots, |l(m-1)n+2n-2-n+1 - l(m-1)n+2n+2-n-2| \}$$

$$= \{ l(m-1)n+3, l(m-1)n+6, \cdots, l_{mn-1} \}.$$

For $s = \frac{n-1}{3}$, Let

$$E_7 = \{ f_1(u_{m,j}u_{m,j+1}) : 3s - 2 \leq j \leq 3s - 1 \}$$

$$= \{ |f(u_{m,j}) - f(u_{m,j+1})| : 3s - 2 \leq j \leq 3s - 1 \}$$

$$= \{ |l(m-1)n+2n-6+2-n+1 - l(m-1)n+2n+4+2-n+1|, \}$$

$$\quad \quad \quad |l(m-1)n+2n-4-2-n+1 - l(m-1)n+2n-2-n+1| \}$$

$$= \{ |l_{mn-3} - l_{mn-1}|, |l_{mn-1} - l_{mn+1}| \} = \{ l_{mn-2}, l_{mn} \}.$$
Now, \( E = \bigcup_{i=1}^{7} E_i = \{l_1, l_2, \ldots, l_{mn}\} \). For \( n \equiv 2(\text{mod } 3) \) and integers \( s = 1, 2, \ldots, \frac{n-2}{3} \),

\[
E'_1 = \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s - 2 \leq j \leq 3s - 1\} \\
= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s - 2 \leq j \leq 3s - 1\} \\
= \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{(m-1)n+2j+2-3s} - l_{(m-1)n+2j+4-3s}| : 3s - 2 \leq j \leq 3s - 1\} \\
= \bigcup_{s=1}^{\frac{n-2}{3}} \{l_{(m-1)n+2j+3-3s} : 3s - 2 \leq j \leq 3s - 1\} \\
= \{l_{(m-1)n+2}, l_{(m-1)n+4}, l_{(m-1)n+5}, l_{(m-1)n+7}, \ldots, l_{mn-3}, l_{mn-1}\}
\]

We find the edge labeling between the end vertex of \( s^{th} \) loop and the starting vertex of \( s+1^{th} \) loop for integers \( s = 1, 2, \ldots, \frac{n-2}{3} \). Let

\[
E'_2 = \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_{m,j}u_{m,j+1})j = 3s\} = \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\
= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \ldots, |f(u_{m,n-2}) - f(u_{m,n-1})|\} \\
= \{|l_{(m-1)n+8-3} - l_{(m-1)n+10-6}|, |l_{(m-1)n+14-6} - l_{(m-1)n+16-9}|, \ldots, |l_{(m-1)n+2n-2-n+2} - l_{(m-1)n+2n-2-n-1}|\} \\
= \{|l_{(m-1)n+5} - l_{(m-1)n+4}|, |l_{(m-1)n+8} - l_{(m-1)n+7}|, \ldots, |l_{(m-1)n+n} - l_{(m-1)n+n-1}|\} \\
= \{l_{(m-1)n+3}, l_{(m-1)n+6}, \ldots, l_{mn-2}\}
\]

For \( s = \frac{n+1}{3} \), let

\[
E'_3 = \{f_1(u_{m,j}u_{m,j+1}) : j = 3s - 2\} = \{|f(u_{m,j}) - f(u_{m,j+1})| : j = n-1\} \\
= \{|f(u_{m,n-1}) - f(u_{m,n})|\} = \{|l_{(m-1)n+2n-n-1} - l_{(m-1)n+2n-2-n-1}|\} \\
= \{|l_{mn-1} - l_{mn+1}|\} = \{l_{mn}\}.
\]

Therefore, \( E' = \bigcup_{i=1}^{3} E'_i \). Let

\[
E_0 = \{f_1(u_0v_1)\} = \{|f(u_0) - f(v_1)|\} = \{|l_0 - l_{mn+2}|\} = \{l_{mn+2}\}.
\]
For $t \equiv 2 \pmod{3}$ and $r = 1, 2, \cdots, \frac{t-2}{3}$, let

$$E_1^r = \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : 3r - 2 \leq k \leq 3r - 1\}$$

$$= \bigcup_{r=1}^{\frac{t-2}{3}} \{[f(v_k) - f(v_{k+1})] : 3r - 2 \leq k \leq 3r - 1\}$$

$$= \{[f(v_1) - f(v_2)], [f(v_2) - f(v_3)], \ldots, [f(v_{t-3}) - f(v_{t-2})]\}$$

$$\cdots \cup \{[f(v_{t-4}) - f(v_{t-3})], [f(v_{t-3}) - f(v_{t-2})]\}$$

$$= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\}$$

$$\cup \{|l_{mn+8+3-6} - l_{mn+10+3-6}|, |l_{mn+10+3-6} - l_{mn+12+3-6}|\}$$

$$\cdots \cup \{|l_{mn+3+2t-8-t+2} - l_{mn+3+2t-6-t+2}|, |l_{mn+3+2t-6-t+2} - l_{mn+3+2t-4-t+2}|\}$$

$$= \{|l_{mn+2} - l_{mn+4} , |l_{mn+4} - l_{mn+6}|\} \cup \{|l_{mn+5} - l_{mn+7} |, |l_{mn+7} = l_{mn+9}|\}$$

$$\cdots \cup \{|l_{mn+t-3} - l_{mn+t-1} |, |l_{mn+t-1} - l_{mn+t+1}|\}$$

$$= \{|l_{mn+3}, l_{mn+5}\} \cup \{|l_{mn+6}, l_{mn+8}|\} \cup \cdots \cup \{|l_{mn+t-2}, l_{mn+t}|\}$$

$$= \{|l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-2}, l_{mn+t}|.$$  

We find the edge labeling between the end vertex of $r^{th}$ loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \cdots, \frac{t-2}{3}$. Let

$$E_2^r = \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} = \bigcup_{r=1}^{\frac{t-2}{3}} \{[f(v_k) - f(v_{k+1})] : k = 3r\}$$

$$= \{[f(v_3) - f(v_4)], [f(v_6) - f(v_7)], \ldots, [f(v_{t-1}) - f(v_t)]\}$$

$$= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, |l_{mn+3+12-6} - l_{mn+3+14-9}|,$$

$$\cdots, |l_{mn+3+2t-4-t+2} - l_{mn+3+2t-2-t+1}|\}$$

$$= \{|l_{mn+6} - l_{mn+5} , |l_{mn+9} - l_{mn+8}|, |l_{mn+t+1} - l_{mn+t}|\}$$

$$= \{|l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t-1}|.$$  

For $s = \frac{t+1}{3}$, let

$$E_3^r = \{f_1(v_k v_{k+1}) : k = 3r - 2\} = \{[f(v_k) - f(v_{k+1})] : k = 3r - 2\}$$

$$= \{|l_{mn+3+2t-2-t-1} - l_{mn+3+2t-2t-1}|\} = \{|l_{mn+t} - l_{mn+t+2}| = \{l_{mn+t+1}\}$$

Therefore, $E^r = E_0 \cup E_1^r \cup E_2^r \cup E_3^r = \{l_{mn+2}, l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-2}, l_{mn+t}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t-1}\}$. Now, $E \cup E^r = \bigcup_{i=1}^{7} E_i \cup E_0 \cup E_1^r \cup E_2^r \cup E_3^r = \{l_1, l_2, \cdots, l_{mn}, l_{mn+2}, l_{mn+3}, l_{mn+4}, \cdots, l_{mn+t-2}, l_{mn+t-1}, l_{mn+t+1}\}$. So, the edge labels of $G$ are
distinct. For \( t \equiv 1(\text{mod } 3) \) and integers \( r = 1, 2, \cdots, \frac{t-1}{3} \), let

\[
E''_1 = \bigcup_{r=1}^{\frac{t-1}{3}} \{ f_1(v_kv_{k+1}) : 3r - 2 \leq k \leq 3r - 1 \}
\]

\[
= \bigcup_{r=1}^{\frac{t-1}{3}} \{ |f(v_k) - f(v_{k+1})| : 3r - 2 \leq k \leq 3r - 1 \}
\]

\[
= \{ |f(v_1) - f(v_2)|, |f(v_2) - f(v_3)| \} \cup \{ |f(v_4) - f(v_5)|, |f(v_5) - f(v_6)| \} \cup 
\]

\[
\cdots \cup \{ |f(v_{t-3}) - f(v_{t-2})|, |f(v_{t-2}) - f(v_{t-1})| \}
\]

\[
= \{ |l_{mn+3+2+3} - l_{mn+3+4+3}|, |l_{mn+3+4+3} - l_{mn+3+6+3}| \}
\]

\[
\cup \{ |l_{mn+3+8+6} - l_{mn+3+10-6}|, |l_{mn+3+10-6} - l_{mn+3+12-6}| \}
\]

\[
\cdots \cup \{ |l_{mn+3+2t-6-t+1} - l_{mn+3+2t-4-t+1}|, |l_{mn+3+2t-4-t+1} - l_{mn+3+2t-2-t+1}| \}
\]

\[
= \{ |l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}| \} \cup \{ |l_{mn+5} - l_{mn+7}|, |l_{mn+7} - l_{mn+9}| \} \cup 
\]

\[
\cdots \cup \{ |l_{mn+t-2} - l_{mn+t}|, |l_{mn+t} - l_{mn+t+2}| \}
\]

\[
= \{ l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-1}, l_{mn+t+1} \}.
\]

We find the edge labeling between the end vertex of \( r^{th} \) loop and the starting vertex of \( (r+1)^{th} \) loop for integers \( r = 1, 2, \cdots, \frac{t-1}{3} \). Let

\[
E''_2 = \bigcup_{r=1}^{\frac{t-1}{3}} \{ f_1(v_kv_{k+1}) : k = 3r \}
\]

\[
= \bigcup_{r=1}^{\frac{t-1}{3}} \{ |f(v_k) - f(v_{k+1})| : k = 3r \}
\]

\[
= \{ |f(v_1) - f(v_4)|, |f(v_4) - f(v_7)|, \cdots, |f(v_{t-1}) - f(v_t)| \}
\]

\[
= \{ |l_{mn+3+6-3} - l_{mn+3+8-6}|, \cdots, |l_{mn+3+2t-2-t+1} - l_{mn+3+2t-2-t}| \}
\]

\[
= \{ |l_{mn+6} - l_{mn+9}|, |l_{mn+9} - l_{mn+12}|, \cdots, |l_{mn+t+2} - l_{mn+t+1}| \}
\]

\[
= \{ l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t} \}
\]

Therefore \( E'''' = E_0 \cup E''_1 \cup E''_2 \) \( = \{ l_{mn+2}, l_{mn+3}, \cdots, l_{mn+t-1}, l_{mn+t}, l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t+1} \} \). Now, \( E \cup E' \cup E'''' = \bigcup_{i=1}^{4} E_i \)

\[
\bigcup_{i=1}^{3} E_i \cup \{ E_0 \cup E''_1 \cup E''_2 \} = \{ l_{1}, l_{2}, \cdots, l_{mn}, l_{mn+2}, l_{mn+3}, \cdots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1} \}
\]

So, the edge labels of \( G \) are distinct. In both cases, \( f \) is a nearly Lucas graceful labeling. Thus \( G = S_{m,n}@P_t \) is a nearly Lucas graceful graph when \( m \equiv 1(\text{mod } 2), n \equiv 1, 2(\text{mod } 3) \) and \( t \equiv 1, 2, (\text{mod } 3) \).

**Example 3.3** A nearly Lucas graceful labeling of \( S_{5,7}@P_7 \) is shown in Fig.3.1.
Theorem 3.4 $C_n$ is a nearly Lucas graceful graph when $n \equiv 1, 2 (\text{mod } 3)$.

Proof Let $G = C_n$ with $V(G) = \{u_i : 1 \leq i \leq n\}$. Let $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ be the edge set of $G$. So, $|V(G)| = n$ and $|E(G)| = n$.

Case 1 $n \equiv 1 (\text{mod } 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $a \in \mathbb{N}$ by $f(u_1) = l_0$. For $s = 1, 2, \cdots, \frac{n-4}{3}$, $f(u_i) = l_{2i-3s}$, $3s - 1 \leq i \leq 3s + 1$ and for $s = \frac{n-1}{3}$, $f(u_i) = l_{2i-3s}$, $3s - 1 \leq i \leq 3s$. We claim that the edge labels are distinct. Let

$$E_1 = \{f_1(u_1u_2), f_1(u_nu_1)\} = \{|f(u_1) - f(u_2)|, |f(u_n) - f(u_1)|\}$$

$$= \{|l_0 - l_1|, |l_{2n-1} - l_0|\} = \{l_1, l_{n+1}\}.$$

For $s = 1, 2, \cdots, \frac{n-1}{3}$, let

$$E_2 = \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_{i}u_{i+1}) \text{ if } 3s - 1 \leq i \leq 3s\}$$

$$= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_i) - f(u_{i+1})| : 3s - 1 \leq i \leq 3s\}$$

$$= \{|f(u_2) - f(u_3), f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \cdots \bigcup \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\}$$

$$= \{|l_1 - l_2|, |l_3 - l_4|\} \bigcup \{|l_4 - l_5|, |l_5 - l_6|\} \bigcup \cdots \bigcup \{|l_{2n-4} - l_{2n-3}|, |l_{2n-2} - l_{2n-1}|\}$$

$$= \{l_2, l_4 \bigcup \{l_5, l_7\} \bigcup \{l_{n-2}, l_n\}.$$ 

We find the edge labeling between the end vertex of $s^{th}$ loop and the starting vertex of $(s+1)^{th}$
loop for integers \( s = 1, 2, \ldots, \frac{n - 1}{3} - 1 \). Let

\[
E_3 = \bigcup_{s=1}^{\frac{n-4}{3}} \{ f_1(u_i u_{i+1}) : i = 3s + 1 \}
\]

\[
= \bigcup_{s=1}^{\frac{n-4}{3}} \{ |f(u_i) - f(u_{i+1})| : i = 3s + 1 \}
\]

\[
= \{ |f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \ldots, |f(u_{n-3} - f(u_{n-2})| \}
\]

\[
= \{ |l_{8-3} - l_{10-6}|, |l_{14-6} - l_{16-9}|, \ldots, |l_{2n-6-n+4} - l_{2n-4-n+1}| \}
\]

\[
= \{ |l_5 - l_4|, |l_8 - l_7|, \ldots, |l_{n-2} - l_{n-3}| \} = \{ l_3, l_6, \ldots, l_{n-4} \}
\]

Now, \( E = \bigcup_{i=1}^{3} E_i = \{ l_1, l_2, l_3, l_4, \ldots, l_{n-2}, l_n, l_{n+1} \} \).

**Case 2** \( n \equiv 2 \pmod{3} \).

Define \( f : V(G) \to \{ l_0, l_1, l_2, \ldots, l_n \} \), \( a \in N \) by \( f(u_1) = l_0, f(u_n) = l_{n+2} \). For \( s = 1, 2, \ldots, \frac{n-2}{3} - 1 \), \( f(u_s) = l_{2s-3s}, 3s-1 \leq i \leq 3s+1 \) and for \( s = \frac{n-2}{3}, f(u_s) = l_{2s-3s}, 3s-1 \leq i \leq 3s \). We claim that the edge labels are distinct. Let

\[
E_1 = \{ f_1(u_1 u_2), f_1(u_{n-1} u_n), f_1(u_n u_1) \}
\]

\[
= \{ |f(u_1) - f(u_2)|, |f(u_{n-1}) - f(u_n)|, |f(u_n) - f(u_1)| \}
\]

\[
= \{ |l_0 - l_1|, |l_{2n-2-n+2} - l_{n+2}|, |l_{n+2} - l_0| \} = \{ l_1, l_{n+1}, l_{n+2} \}
\]

\[
E_2 = \bigcup_{s=1}^{\frac{n-2}{3}} \{ f_1(u_i u_{i+1}) : 3s - 1 \leq i \leq 3s \}
\]

\[
= \bigcup_{s=1}^{\frac{n-2}{3}} \{ |f(u_i) - f(u_{i+1})| : 3s - 1 \leq i \leq 3s \}
\]

\[
= \{ |f(u_2) - f(u_3)|, |f(u_3) - f(u_4)| \} \bigcup \{ |f(u_5) - f(u_6)|, |f(u_6) - f(u_7)| \} \bigcup \ldots \bigcup \{ |f(u_{n-3} - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})| \}
\]

\[
= \{ |l_4 - l_5|, |l_6 - l_7| \} \bigcup \{ |l_{10-6} - l_{12-6}|, |l_{12-6} - l_{14-6}| \} \bigcup \ldots \bigcup \{ |l_{2n-6-n+2} - l_{2n-4-n+2}| \}
\]

\[
= \{ |l_1 - l_2|, |l_3 - l_4| \} \bigcup \{ |l_4 - l_5|, |l_6 - l_7| \} \bigcup \ldots \bigcup \{ |l_{n-4} - l_{n-5}|, |l_{n-2} - l_n| \}
\]

\[
= \{ l_2, l_4, l_5, l_7, \ldots, l_{n-3}, l_{n-1} \}.
\]

We find the edge labeling between the end vertex of \((s-1)th\) loop and the starting vertex of
$s^{th}$ loop for integers $s = 1, 2, \cdots, \frac{n-5}{3}$. Let

$$E_3 = \bigcup_{s=1}^{\frac{n-5}{3}} \{f_1(u_i, u_{i+1}) : i = 3s + 1\}$$

$$= \bigcup_{s=1}^{\frac{n-5}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s + 1\}$$

$$= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{n-4}) - f(u_{n-3})|\}$$

$$= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{2n-8-n+5} - l_{2n-6-n+2}|\} = \{l_3, l_6, \cdots, l_{n-2}\}$$

Now, $E = \bigcup_{i=1}^{3} E_i = \{l_1, l_2, l_3, l_4, \cdots, l_{n-3}, l_{n-2}, l_{n-1}, l_{n+1}, l_{n+2}\}$ So, all these edge labels of $G$ are distinct. In both the cases, $f$ is a nearly Lucas graceful graph. Thus $G = C_n$ is a nearly Lucas graceful graph when $n \equiv 1, 2 (mod\ 3)$. \hfill \Box

**Example 3.5** A nearly Lucas graceful labeling on $C_{13}$ in Case 1 is shown in Fig.3.2.

![Fig.3.2 C_{13}](image1)

**Example 3.6** A nearly Lucas graceful labeling on $C_{14}$ in Case 2 is shown in Fig.3.3.

![Fig.3.3 C_{14}](image2)
References


