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On the mean value of some new sequences

Jiao Chen

Department of Mathematics, Northwest University,
Xi'an, Shaanxi, P.R.China
E-mail: chenjiaogaoling@163.com

Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the Smarandache repetitional sequence, and give two asymptotic formulas for it.

Keywords Euler's summation formula, Abel's identity, Smarandache repetitional sequence.

§1. Introduction

Let k be a fixed positive integer. The famous Smarandache repetitional generalized sequence $S(n,k)$ is defined as following: $n1n, n22n, n333n, n4444n, n55555n, n666666n, n777777$ $7n, n88888888n, n999999999n, n10101010101010101010n, \cdots$. In problem 3 of reference [1], Professor Mihaly Benze asked us to study the arithmetical properties about this sequence. It is interesting for us to study this problem. But it's a pity none had studied it before. At least we haven't seen such a paper yet. In this paper, we shall use the elementary and analytic methods to study the arithmetical properties of the special Smarandache repetitional generalized sequence,and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{k\leq x} S(n,k) = \sum_{i=0}^{m} di(x-i)^3 + fi(2i+1)(x-i)^2 + h\cdot 10^x + O(\frac{100\cdot n}{9}\cdot 10^x) + A.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we give three simple lemmas which are necessary in the proof of our theorem. The proofs of these lemmas can be found in reference [7].

Lemma 1. For any real number $x \geq 1$ and $\alpha > 0$, we have the asymptotic formula

$$\sum_{n\leq x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O\left(x^\alpha\right).$$

Lemma 2. If f has a continuous derivative f' on the interval $[x,y]$, where $0 < y < x$,

$$\sum_{y<k\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

Lemma 3. For any arithmetical function $a(n)$, let $A(x) = \sum_{n\leq x} a(n)$, If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$,

$$\sum_{y<k\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) + \int_y^x A(t)f'(t)dt.$$

Now we use these lemmas to prove our conclusion. First we use the elementary method to obtain an asymptotic formula. To accomplish our theorem easily, we can get the following equations by observing the classification of the Smarandache repetitional sequence.

$S(n, 1) = n \cdot 10^2 + 1 \cdot 10^1 + n \cdot 10^0$

$S(n, 2) = n \cdot 10^3 + 2 \cdot 10^2 + 2 \cdot 10^1 + n \cdot 10^0$

$S(n, 3) = n \cdot 10^4 + 3 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + n \cdot 10^0$

$S(n, 4) = n \cdot 10^5 + 4 \cdot 10^4 + 4 \cdot 10^3 + 4 \cdot 10^2 + 4 \cdot 10^1 + n \cdot 10^0$

$\cdots\cdots$

$S(n, a-2) = n \cdot 10^{a-1} + (a-2) \cdot 10^{a-2} + \cdots + (a-2) \cdot 10^2 + (a-2) \cdot 10^1 + n \cdot 10^0$

$S(n, a-1) = n \cdot 10^{a} + (a-1) \cdot 10^{a-1} + \cdots + (a-1) \cdot 10^2 + (a-1) \cdot 10^1 + n \cdot 10^0$

$S(n, a) = n \cdot 10^{a+1} + a \cdot 10^{a} + \cdots + a \cdot 10^2 + a \cdot 10^1 + n \cdot 10^0.$

Now we estimate the right hand side of the above equations, by lemma 2 we have

$$\begin{aligned}\sum_{k\leq x} S(n,k) &= \sum_{k\leq x} n \cdot [\frac{100 \cdot (10^k - 1)}{9}] + \sum_{k\leq x} \frac{k(k+1)}{2} \cdot 10^1 + \sum_{k\leq x} \frac{(k-1)(k+2)}{2} \cdot 10^2 \\ &\quad + \sum_{k\leq x} \frac{(k-2)(k+3)}{2} \cdot 10^3 + \cdots + \sum_{k\leq x} \frac{[k+(k-2)][k-(k-3)]}{2} \cdot 10^{k-2} \\ &\quad + \sum_{k\leq x} \frac{[k+(k-1)][k-(k-2)]}{2} \cdot 10^{k-1} + k \cdot 10^k. \qquad (1)\end{aligned}$$

Now we estimate the one part of the right hand side of the above equations, by lemma 2 and lemma 3 we have

$$\begin{aligned}\sum_{k\leq x} k(k+1) &= [\frac{x^2}{2} + O(x)](x+1) - 1 - O(2) - \int_1^x [\frac{t^2}{2} + O(t)]dt \\ &= \frac{x^3}{2} + \frac{x^2}{2} - 1 - \int_1^x [\frac{t^2}{2} + O(t)]dt + O(x^2) \\ &= \frac{x^3}{2} + \frac{x^2}{2} - 1 - \frac{t^2}{2}dt + O(\int_1^x tdt) + O(x^2) \\ &= \frac{x^3}{3} + \frac{x^2}{2} + O(x^2) + C1, \qquad (2)\end{aligned}$$

where we have used the identity $C1$.

Similarly, we also have the asymptotic formulae

$$\begin{aligned}
\sum_{k\le x}(k-1)(k+2) &= [\frac{(x-1)^2}{2}+O(x-1)(x+2)-\int_1^x[\frac{(t-1)^2}{2}+O(t-1)]dt \\
&= \frac{(x-1)^2(x+2)}{2}]-\int_1^x\frac{(t-1)^2}{2}dt+\int_1^x O(t-1)dt+O(x-1)^2) \\
&= \frac{(x-1)^2(x-1+3)}{2}-\frac{(x-1)^3}{6}+O(\int_1^x(t-1)dt)+O(x^2) \\
&= \frac{(x-1)^3}{3}+\frac{3(x-1)^2}{2}+O((x-1)^2)+C2, \qquad (3)
\end{aligned}$$

where we have used the identity $C2$. Similarly, we also have the asymptotic formulae

$$\begin{aligned}
\sum_{k\le x}(k-a+1)(k+a) &= [\frac{(x-a+1)^2}{2}+O(x-a+1)](x+a)-\int_1^x[\frac{(t-a+1)^2}{2} \\
&\quad +O(t-a+1)]dt \\
&= \frac{(x-a+1)^2(x+a)}{2}-\int_1^x\frac{(t-a+1)^2}{2}dt+\int_1^x O(t-a+1)dt \\
&\quad +O(x-a+1)^2 \\
&= \frac{(x-a+1)^2(x-a+1+2a-1)}{2}-\frac{(x-a+1)^3}{6} \\
&\quad +O(\int_1^x t-a+1dt) \\
&= \frac{(x-a+1)^3}{3}+\frac{(2a-1)(x-a+1)^2}{2}+O((x-a+1)^2) \\
&\quad +Ca-1. \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\sum_{k\le x}(k-a)(k+a+1) &= [\frac{(x-a)^2}{2}+O(x-a)](x+a+1)-\int_1^x[\frac{(t-a)^2}{2}+O(t-a)]dt \\
&= \frac{(x-a)^2(x+a+1)}{2}-\int_1^x\frac{(t-a)^2}{2}dt+\int_1^x O(t-a)dt+O(x-a)^2 \\
&= \frac{(x-a)^2(x-a+2a+1)}{2}-\frac{(x-a)^3}{6}+O(\int_1^x t-adt) \\
&= \frac{(x-a)^3}{3}+\frac{(2a+1)(x-a)^2}{2}+O((x-a)^2)+Ca. \qquad (5)
\end{aligned}$$

Finally ,we can use the lemma 2 to get the following formulae

$$\begin{aligned}
\sum_{k\le x}k\cdot 10^k &= [\frac{x^2}{2}+O(x)]\cdot 10^x-\int_1^x\ln 10[\frac{t^2}{2}+O(t)]10^t dt \\
&= \frac{x^2}{2}\cdot 10^x+O(x\cdot 10^x)-\frac{1}{2}\int_1^x t^2 d10^t-\int_1^x O(t\cdot 10^t)dt \\
&= \frac{x^2}{2}\cdot 10^x+O(x\cdot 10^x)-\frac{x^2}{2}\cdot 10^x-O(\int_1^x t\cdot 10^t dt) \\
&= \frac{x\cdot 10^x}{\ln 10}-\frac{10^x}{(\ln 10)^2}+O(x\cdot 10^x)+A1 \qquad (6)
\end{aligned}$$

From (1) (2), (3), (4), (5) and (6) we deduce the asymptotic formula

$$\sum_{k\leq x} S(n,k) = \sum_{i=0}^{m} di(x-i)^3 + fi(2i+1)(x-i)^2 + h\cdot 10^x + O(\frac{100\cdot n}{9}\cdot 10^x) + A.$$

Now combining two methods we may immediately deduce our theorem.

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