On the mean value of $SSMP(n)$ and $SIMP(n)$

Yiren Wang
Department of Mathematics, Northwest University, Xi’an, Shaanxi, P.R.China

Abstract The main purpose of this paper is to study the mean value properties of the Smarandache Superior $m$-th power part sequence $SSMP(n)$ and the Smarandache Inferior $m$-th power part sequence $SIMP(n)$, and give several interesting asymptotic formula for them.

Keywords Smarandache Superior $m$-th power part sequence, Smarandache Inferior $m$-th power part sequences, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer $n$, the Smarandache Superior $m$-th power part sequence $SSMP(n)$ is defined as the smallest $m$-th power greater than or equal to $n$. The Smarandache Inferior $m$-th power part sequence $SIMP(n)$ is defined as the largest $m$-th power less than or equal to $n$. For example, if $m = 2$, then the first few terms of $SIMP(n)$ are: 0, 1, 1, 1, 4, 4, 4, 4, 9, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 16, 25, · · ·. The first few terms of $SSMP(n)$ are: 1, 4, 4, 4, 9, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 16, 25, · · ·. If $m = 3$, then The first few terms of $SSMP(n)$ are: 1, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 27, · · ·. The first few terms of $SIMP(n)$ are: 0, 1, 1, 1, 1, 1, 1, 1, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 27, · · ·. Now we let

$$S_n = (SSMP(1) + SSMP(2) + · · · + SSMP(n))/n;$$
$$I_n = (SIMP(1) + SIMP(2) + · · · + SIMP(n))/n;$$
$$K_n = \sqrt[n]{SSMP(1) + SSMP(2) + · · · + SSMP(n)};$$
$$I_n = \sqrt[n]{SIMP(1) + SIMP(2) + · · · + SIMP(n)}.$$

In reference [2], Dr. K.Kashihara asked us to study the properties of these sequences. Gou Su [3] studied these problem, and proved the following conclusion:

For any real number $x > 2$ and integer $m = 2$, we have the asymptotic formula

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{\frac{5}{2}}\right), \quad \sum_{n \leq x} SIMP(n) = \frac{x^2}{2} + O\left(x^{\frac{5}{2}}\right),$$

and

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{2}}\right), \quad \lim_{n \to \infty} \frac{S_n}{I_n} = 1.$$

This work is supported by the Shaanxi Provincial Education Department Foundation 08JK433.
In this paper, we shall use the elementary method to give a general conclusion. That is, we shall prove the following:

**Theorem 1.** Let \( m \geq 2 \) be an integer, then for any real number \( x > 1 \), we have the asymptotic formula
\[
\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right),
\]
and
\[
\sum_{n \leq x} SIMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right).
\]

**Theorem 2.** For any fixed positive integer \( m \geq 2 \) and any positive integer \( n \), we have the asymptotic formula
\[
S_n - I_n = \frac{m(m-1)}{2m-1}n^{1-\frac{1}{m}} + O\left(n^{1-\frac{1}{m}}\right).
\]

**Corollary 1.** For any positive integer \( n \), we have the asymptotic formula
\[
\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{m}}\right),
\]
and the limit \( \lim_{n \to \infty} \frac{S_n}{I_n} = 1 \).

**Corollary 2.** For any positive integer \( n \), we have the asymptotic formula
\[
\frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right),
\]
and the limit \( \lim_{n \to \infty} \frac{K_n}{L_n} = 1 \), \( \lim_{n \to \infty} (K_n - L_n) = 0 \).

§2. Proof of the theorems

In this section, we shall use the Euler summation formula and the elementary method to complete the proof of our Theorems. For any real number \( x > 2 \), it is clear that there exists one and only one positive integer \( M \) satisfying \( M^m < x \leq (M + 1)^m \). That is, \( M = \lfloor x^{\frac{1}{m}} \rfloor + O(1) \).

So we have
\[
\sum_{n \leq x} SSMP(n) = \sum_{n \leq M^m} SSMP(n) + \sum_{M^m < n \leq x} SSMP(n)
\]
\[
= \sum_{k \leq M} (k^m - (k - 1)^m)k^m + ([x] - (M^m + 1))(M + 1)^m
\]
\[
= \sum_{k \leq M} (mk^{2m-1} + O(k^{2m-2})) + ([x] - M^m - 1)(M + 1)^m
\]
\[
= \frac{m \cdot M^{2m}}{2m} + O\left(M^{2m-1}\right) + ([x] - M^m - 1)(M + 1)^m
\]
\[
= \frac{M^{2m}}{2} + O\left(M^{2m-1}\right).
\]
Note that $M = x^{\frac{1}{m}} + O(1)$, from the above estimate we have the asymptotic formula

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2 - \frac{1}{m}}\right).$$

This proves the first formula of Theorem 1.

Now we prove the second one. For any real number $x > 1$, we also have

$$\sum_{n \leq x} SIMP(n) = \sum_{n < M} SIMP(n) + \sum_{M \leq n \leq x} SIMP(n)$$

$$= \sum_{k \leq M} (k^m - (k - 1)^m)(k - 1)^m + \sum_{M \leq n \leq x} M^m$$

$$= \sum_{k \leq M} (mk^{2m-1} + O(k^{2m-2})) + ([x] - M^m + 1) M^m$$

$$= \frac{M^{2m}}{2} + O\left(M^{2m-1}\right) + ([x] - M^m + 1) M^m.$$

Note that

$$(\lceil x \rceil - M^m + 1) M^m \leq M^{2m-1} \leq x^{1 - \frac{1}{m}}.$$ 

Therefore,

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2 - \frac{1}{m}}\right).$$

This completes the proof of Theorem 1.

To prove Theorem 2, let $x = n$, then from the method of proving Theorem 1 we have

$$S_n - I_n = \frac{1}{n} (SSMP(1) + SSMP(2) + \cdots + SSMP(n))$$

$$- \frac{1}{n} (SIMP(1) + SIMP(2) + \cdots + SIMP(n))$$

$$= \frac{1}{n} \left( \sum_{k \leq M} (k^m - (k - 1)^m)k^m + ([x] - (M^m + 1))(M + 1)^m \right)$$

$$- \frac{1}{n} \left( \sum_{k \leq M} (k^m - (k - 1)^m)(k - 1)^m + ([x] - M^m + 1) M^m \right)$$

$$= \frac{1}{n} \sum_{k \leq M} m(m - 1)k^{2m-2} + O\left(\frac{1}{n} M^{2m-2}\right)$$

$$= \frac{m(m - 1)}{n(2m - 1)} M^{2m-1} + O\left(\frac{1}{n} M^{2m-2}\right).$$

Note that $M^m < n \leq (M+1)^m$ or $M = n^{\frac{1}{m}} + O(1)$, from the above formula we may immediately deduce that

$$S_n - I_n = \frac{m(m - 1)}{2m - 1} n^{1 - \frac{1}{m}} + O\left(n^{1 - \frac{2}{m}}\right).$$

This completes the proof of Theorem 2.
Now we prove the Corollaries. Note that the asymptotic formula

\[ I_n = \frac{1}{n} (\text{SIMP}(1) + \text{SIMP}(2) + \cdots + \text{SIMP}(n)) = \frac{1}{n} \left( \frac{n^2}{2} + O\left(\frac{n^{2m-1}}{m}\right) \right) = \frac{n}{2} + O\left(n^{1-\frac{m}{n}}\right) \]

and

\[ S_n = \frac{1}{n} (\text{SSMP}(1) + \text{SSMP}(2) + \cdots + \text{SSMP}(n)) = \frac{1}{n} \left( \frac{n^2}{2} + O\left(\frac{n^{2m-1}}{m}\right) \right) = \frac{n}{2} + O\left(n^{1-\frac{m}{2}}\right). \]

From the above two formula we have

\[ \frac{S_n}{I_n} = \frac{n}{2} + O\left(\frac{n^{m-\frac{1}{m}}}{m}\right) = 1 + O\left(n^{-\frac{1}{m}}\right). \]

Therefore, we have the limit formula

\[ \lim_{n \to \infty} \frac{S_n}{I_n} = 1. \]

Using the same method we can also deduce that

\[ K_n = \sqrt[n]{\text{SSMP}(1) + \text{SSMP}(2) + \cdots + \text{SSMP}(n)} = \left( \frac{n^2}{2} + O\left(\frac{n^{2m-1}}{m}\right) \right)^{\frac{1}{n}} \]

and

\[ L_n = \sqrt[n]{\text{SIMP}(1) + \text{SIMP}(2) + \cdots + \text{SIMP}(n)} = \left( \frac{n^2}{2} + O\left(\frac{n^{2m-1}}{m}\right) \right)^{\frac{1}{n}} \]

From these formula we may immediately deduce that

\[ \frac{K_n}{L_n} = \left( \frac{n^2}{2} + O\left(\frac{n^{2m-1}}{m}\right) \right)^{\frac{1}{2}} = \left(1 + O\left(n^{-\frac{1}{m}}\right)\right)^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right). \]

Therefore, we have the limit formula

\[ \lim_{n \to \infty} \frac{K_n}{L_n} = 1. \]

Note that \( \lim_{n \to \infty} K_n = \lim_{n \to \infty} L_n = 1, \) we may immediately deduce that

\[ \lim_{n \to \infty} (K_n - L_n) = 0. \]

This completes the proof of Corollary 2.

References


